高数基础

一、极限与连限

Part 1 极限

一. defs

1. 极限

case 1. 若 $\forall \varepsilon>0$，$\exists N>0$，当 $n>N$ 时，$|a_n-A|<\varepsilon$

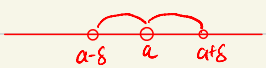
则有 $\lim\limits_{x\to\infty} a_n = A$

Notes:

① $x\to a$ $\begin{cases} x\neq a \\ x\to a^+ \cdot x\to a^- \end{cases}$

② $\lim\limits_{x\to a} f(x)$ 与 $f(a)$ 无关

③ $0<|x-a|<\delta$



Case 2 (ε-δ)

若 $f(x)$ 在 $x=a$ 的去心邻域内有定义

If $\forall \varepsilon>0$, $\exists \delta>0$ 当 $0<|x-a|<\delta$ 时

有 $|f(x)-A|<\varepsilon$

即 $\lim\limits_{x\to a} f(x)=A$

Notes

① If $\forall \varepsilon>0$, $\exists \delta>0$, 当 $x\in(a-\delta, a)$ 时, $|f(x)-A|<\varepsilon$

即 $\lim\limits_{x\to a^-} f(x)=A$

If $\forall \varepsilon>0$, $\exists \delta>0$, 当 $x\in(a, a+\delta)$ 时, $|f(x)-B|<\varepsilon$

即 $\lim\limits_{x\to a^+} f(x)=B$

☆ Point 1: $\lim\limits_{x\to a} f(x)\exists \iff f(a-0)$, $f(a+0)$ $\exists$ 且相等

☆⑤ Point 2.

对于含有 $a^{\frac{?}{x-b}}$ 或 $a^{\frac{?}{bx}}$ 的 $f(x)$

当 $x\to b$ 时 需讨论 $x\to b^+$ $x\to b$

例:

$$f(x)=\frac{e^{\frac{1}{x-1}}}{1+e^{\frac{1}{x-1}}},\quad \lim_{x\to 1} f(x) 是否\exists ?$$

解:

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{0}{1}=0$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{e^{\frac{1}{x-1}}}{e^{\frac{1}{x-1}}}=1$$

$\lim\limits_{x\to 1^-} f(x)\neq \lim\limits_{x\to 1^+} f(x)$ 故 $\lim\limits_{x\to 1} f(x)$ 不$\exists$

Case 3 $(\xi - X)$ $\begin{cases} x \to +\infty \\ x \to \infty \\ x \to -\infty \end{cases}$

若 $\forall \xi > 0$，$\exists X > 0$，当 $x > X$ 时，$|f(x) - A| < \xi$

即 $\lim\limits_{x \to +\infty} f(x) = A$

无穷小

① defs：If $\lim\limits_{x \to a} f(x) = 0$，称 $f(x)$ 当 $x \to a$ 时无穷小

② 层次

Case 1 $\lim \frac{\beta}{\alpha} = 0$，$\beta = o(\alpha)$

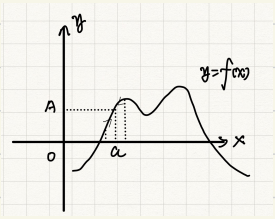
Case 2 $\lim \frac{\beta}{\alpha} = K$ $(\neq 0, \infty) \Rightarrow \beta = O(\alpha)$

若 $K = 1 \Rightarrow \beta \sim \alpha$

二. 性质

1. 唯一性：极限存在必唯一

2. 保号性



若 $\lim_{x\to a} f(x)=A$ $\begin{cases}>0\\<0\end{cases}$

$\exists \delta>0$ 当 $0<|x-a|<\delta$ $f(x)$ $\begin{cases}>0\\<0\end{cases}$

例 1. $f(0)=1$，且 $\lim_{x\to 0}\frac{f(x)-1}{x^2}=-1$，$x=0$ 是否为极值点

解：$\because \lim_{x\to 0}\frac{f(x)-1}{x^2}=-1<0$

$\therefore \exists \delta$，$0<|x|<\delta$ 时，$\frac{f(x)-1}{x^2}<0$

$\therefore f(x)-1<0 \Rightarrow f(x)<1=f(0)$

即 $x=0$ 为极大值点

例 2 $f'(1)=0$，且 $\lim\limits_{x\to 1}\dfrac{f(x)}{\sin\pi x}=1$，$x=1$ 是否为极值点？

解：

$$\lim_{x\to 1}\frac{f(x)}{\sin\pi x}=1>0$$

则 $\exists\delta$，$0<|x-1|<\delta$ 时，有 $\dfrac{f(x)}{\sin\pi x}>0$

当 $x\in(1,1+\delta)$ 时 $\sin\pi x<0$ $f'(x)<0$

$x\in(1-\delta,1)$ 时 $\sin\pi x>0$ $f'(x)>0$

$\therefore f(x)$ 先 ↗，后 ↘

故 $x=1$ 为极大值点

极限存在准则

准则 I：夹逼定理

数列型为例 If $\begin{cases} ① \ a_n \le b_n \le c_n \\ ② \ \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = A \end{cases}$

则 $\lim\limits_{n\to\infty} b_n = A$

例 1.

$$\lim_{n\to\infty} (2^n + 3^n + 4^n)^{\frac{1}{n}}$$

解：$4^n \le 2^n + 3^n + 4^n \le 3 \cdot 4^n$

$$4 \le (2^n + 3^n + 4^n)^{\frac{1}{n}} \le 3^{\frac{1}{n}} \cdot 4$$

$\lim\limits_{n\to\infty} 3^{\frac{1}{n}} \cdot 4 = 4 \cdot 1 = 4$

即 原式极限为 4.

例 2　$f(x)=\lim\limits_{n\to\infty}\sqrt[n]{1+x^n+(\frac{x^2}{2})^n}$　$(x>0)$　求 $f(x)$

解：$\lim\limits_{n\to\infty}\left[1+x^n+(\frac{x^2}{2})^n\right]^{\frac{1}{n}}$

$=\max\left\{1,\ x,\ \frac{x^2}{2}\right\}=\begin{cases}1 & 0<x\le 1\\ x & 1<x\le 2\\ \frac{x^2}{2} & x>2\end{cases}$

型二．n 项和极限．

① 先和后积型

$$\lim_{n\to\infty}\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(2n-1)(2n+1)}\right]$$

解：$=\lim\limits_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{2n-1}-\frac{1}{2n+1}\right]$

$=\lim\limits_{n\to\infty}\frac{1}{2}\frac{2n}{2n+1}=\frac{1}{2}$

② 分子或分母次数不齐 —— 夹逼定理

例1

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n+1}}+\cdots+\frac{1}{\sqrt{4n^2+n}}\right)$$

解：设 $\left(\frac{1}{\sqrt{4n+1}}+\cdots+\frac{1}{\sqrt{4n^2+n}}\right)\triangleq b_n$

$$\because \frac{1}{\sqrt{4n^2+n}}\le\frac{1}{\sqrt{4n^2+i}}\le\frac{1}{\sqrt{4n^2+1}}$$

$$\therefore \frac{n}{\sqrt{4n^2+n}}\le b_n\le\frac{n}{\sqrt{4n^2+1}}$$

$$\lim_{n\to\infty}\frac{1}{\sqrt{4+\frac{1}{n}}}=1/2 \qquad \lim_{n\to\infty}\frac{1}{\sqrt{4+\frac{1}{n^2}}}=\frac{1}{2}$$

$$\therefore \lim_{n\to\infty} b_n = 1/2$$

例2. $\lim_{n\to\infty}\left(\frac{1}{n^2+1}+\cdots+\frac{n}{n^2+n}\right)$

解：设 $\frac{1}{n^2+1}+\cdots+\frac{n}{n^2+n}\triangleq b_n$

$$\because \frac{i}{n^2+n+1}\le\frac{i}{n^2+n+i}\le\frac{i}{n^2+n+n}$$

$$\therefore \frac{\frac{1}{2}(n+1)n}{n^2+n+1}\le b_n\le\frac{\frac{1}{2}(n+1)n}{n^2+2n}$$

$$\therefore \lim_{n\to\infty}\frac{\frac{1}{2}(n+1)n}{n^2+n+1}=\lim_{n\to\infty}\frac{\frac{1}{2}(n+1)n}{n^2+2n}=1/2$$

即 $\lim_{n\to\infty} b_n = 1/2$

③ 分子次数齐，分母次数齐
分母比分子多一次
$\Rightarrow$ 定积分定义

记 $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f(x)dx$

例1 $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\cdots+\frac{1}{n+n})$

解：原式 $=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{n+i}$

$=\frac{1}{n}\lim\limits_{n\to\infty}\frac{1}{1+\frac{i}{n}}=\int_0^1\frac{1}{1+x}dx$

$=\ln 2$

例2 $\lim\limits_{n\to\infty}(\frac{n}{n^2+1^2}+\cdots+\frac{n}{n^2+n^2})$

解：$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{n}{n^2+i^2}$

$=\frac{1}{n}\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\frac{1}{1+\frac{i^2}{n^2}}=\int_0^1\frac{1}{1+x^2}dx$

$=\arctan x\Big|_0^1=\pi/2$

准则 II. 单调有界数列必有极限

Note:

① 有界: If $\exists M>0$, $\forall n$, 总有 $|a_n|<M$, 称 $\{a_n\}$ 有界

If $\exists M_1$, $\forall n$, $a_n \geq M_1$, $\{a_n\}$ 有下界

If $\exists M_2$, $\forall n$ $a_n \leq M_2$, $\{a_n\}$ 有上界.

☆Point: a_n 有界 $\Longleftrightarrow$ 有上, 下界

② $\{a_n\}\uparrow$ 无上界 $\Rightarrow \lim\limits_{n\to\infty} a_n = +\infty$

$\{a_n\}\uparrow$ $a_n \leq M \Rightarrow \lim\limits_{n\to\infty} a_n$ $\exists$

$\{a_n\}\downarrow$ 无下界 $\Rightarrow \lim\limits_{n\to\infty} a_n = -\infty$

$\{a_n\}\downarrow$ $a_n \geq M \Rightarrow \lim\limits_{n\to\infty} a_n$ $\exists$

③ Point 重要不等式 1) $x\geq 0$ $\sin x \leq x$

2) $x>0$ $\ln(x+1) < x$

题目中有 函数和 $\Rightarrow$ 上, 下界 m, M

$\exists\, \xi \in [a,b]$

型二　$\lim_{n\to\infty} a_n$ ∃ 问题

例 $a_1>0$，且 $a_{n+1}=\ln(a_n+1)$　$(n=1,2,\cdots)$
证：a_n 的极限存在，并求出 $\lim_{n\to\infty} a_n$

解：单调有界

1° $a_1>0$，设 $a_k>0$　∵ $a_{k+1}=\ln(1+a_k)>0$
∴ 对 $\forall n$，$a_n>0$

2° ∵ $a_n>0$　∴ $a_{n+1}=\ln(1+a_n)<a_n$　∴ $a_n\downarrow$
即 a_n 极限存在

3° 令 $\lim_{n\to\infty} a_n=A$　∴ $A=\ln(1+A)\Rightarrow A=0$
即 $\lim_{n\to\infty} a_n=A$

例 2. 设 $a_1=\sqrt{2}$，$a_2=\sqrt{2+\sqrt{2}}$，$a_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$ ···
证明 $\lim_{n\to\infty} a_n$ 存在，并求之

解：1° $a_{n+1}=\sqrt{2+a_n}$

2° $a_1<a_2$　设 $a_k<a_{k+1}$
则 $2+a_k<2+a_{k+1}\Rightarrow\sqrt{2+a_k}<\sqrt{2+a_{k+1}}$
$\Rightarrow a_{k+1}<a_{k+2}$
对 $\forall n$，有 $a_n<a_{n+1}$　∴ $a_n\uparrow$

3° 设 $a_n\le 2$
$a_1=\sqrt{2}\le 2$　设 $a_{k+1}=2$　$a_{k+1}=\sqrt{2+a_k}\le\sqrt{2+2}=2$
∴ $\lim_{n\to\infty} a_n$ 存在　$A=\sqrt{2+A}$　∴ $A=2$

例 3. 设 $a_1=2$ $a_{n+1}=\frac{1}{2}(a_n+\frac{1}{a_n})$ $(n=1,2,\cdots)$

证 $\lim_{n\to\infty} a_n$ $\exists$

解: 1° $a_{n+1}\geq \frac{1}{2}\times 2=1$ $(a_n+\frac{1}{a_n}\geq 2)$

2° $a_{n+1}-a_n=\frac{1}{2}(\frac{1}{a_n}-a_n)=\frac{1-a_n^2}{2a_n}\leq 0$

$\therefore a_n\downarrow$ $\therefore \lim_{n\to\infty} a_n$ $\exists$

(四) 无穷小性质

1. 一般性质

① $|\alpha|\leq M$, $\beta\to 0$, $\Rightarrow \alpha\beta\to 0$

② $\lim f(x)=A \Leftrightarrow f(x)=A+\alpha$ $(\alpha\to 0)$

2. 等价性质

① $\alpha\sim\alpha_1$, $\beta\sim\beta_1$ 且 $\lim\frac{\beta_1}{\alpha_1}=A \Rightarrow \lim\frac{\beta}{\alpha}=A$

② $x\to 0$ 时

i) $x\sim\sin x\sim\tan x\sim\arcsin x\sim\arctan x$
$\sim e^x-1\sim\ln(1+x)$

ii) $1-\cos x\sim\frac{1}{2}x^2$

iii) $(1+x)^a-1\sim ax$

3. 重要极限

当 $x \in (0, \frac{\pi}{2})$ $\quad \sin x < x < \tan x$

(I) $\lim\limits_{\Delta \to 0} \frac{\sin \Delta}{\Delta} = 1$

(II) $\lim\limits_{\Delta \to 0} (1+\Delta)^{\frac{1}{\Delta}} =$

型三 不定型

- $\frac{0}{0}$, 1^{∞} ☆
- $\frac{\infty}{\infty}$
- $0 \cdot \infty$, $\infty - \infty$, ∞^0, 0^0

Case 1 $\frac{0}{0}$

① 三大习惯

- $u(x)^{v(x)} \Rightarrow e^{v(x)\ln u(x)}$
- $\ln(\quad) = \ln(1+\Delta)$ $(\Delta \to 0)$
- $(\quad) - 1 =$
  - $e^{\Delta} - 1 \sim \Delta$
  - $(1+\Delta)^{a} - 1 \sim a\Delta$ $(\Delta \to 0)$

② 常见

- $x - \ln(1+x) \sim \frac{1}{2}x^2$
- x, $\sin x$, $\tan x$, $\arctan x$, $\arcsin x$ 任二者差为三阶无穷小

③ 误区：精确度问题

常见于分数型极限，保证分子与分母的精度一致

如 $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$ 此时 $x-\sin x \sim x-x=0$ (X)

正解为 $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{x-[x-\frac{1}{3!}x^3+o(x^3)]}{x^3}=\dfrac{1}{6}$

例 1.

$$\lim_{x\to 0}\frac{(1+2x)^{\sin x}-1}{x^2}$$

解：$\lim\limits_{x\to 0}\dfrac{e^{\sin x\ln(1+2x)}-1}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{\sin x\ln(1+2x)}{x^2}=\lim\limits_{x\to 0}\dfrac{2x^2}{x^2}=2$

例 2.

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$$

解：$\lim\limits_{x\to 0}\dfrac{\tan x(1-\cos x)}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{x\cdot\frac{1}{2}x^2}{x^3}=\dfrac{1}{2}$

例 3.

$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+x}}{x^3}$$

解：$\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3(\sqrt{1+\tan x}+\sqrt{1+x})}$

$=\lim\limits_{x\to 0}\dfrac{1}{2}\cdot\dfrac{\tan x-x}{x^3}$

$=\dfrac{1}{3}$

例 4. $\lim\limits_{x\to 0}\dfrac{e^{\arcsin x}-e^{x}}{x^3}$

解: $\lim\limits_{x\to 0}\dfrac{e^{x}\left(e^{\arcsin x-x}-1\right)}{x^3}$

$=\lim\limits_{x\to 0}e^{x}\cdot\dfrac{\arcsin x-x}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{\arcsin x-x}{x^3}=\lim\limits_{x\to 0}\dfrac{(1-x^2)^{-\frac{1}{2}}-1}{3x^2}=\dfrac{1}{6}$

例 5

$\lim\limits_{x\to 0}\dfrac{\ln\frac{x}{\sin x}}{x^2}$

解: $\lim\limits_{x\to 0}\dfrac{\ln\left(1+\frac{x}{\sin x}-1\right)}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{\frac{x}{\sin x}-1}{x^2}=\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\dfrac{1}{6}$

例 6.

$\lim\limits_{x\to 0}\dfrac{\left(\frac{1+\cos x}{2}\right)^{x}-1}{x^3}$

解: $\lim\limits_{x\to 0}\dfrac{e^{x\ln\left(\frac{1+\cos x}{2}\right)}-1}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{x\ln\left(\frac{1+\cos x}{2}\right)}{x^3}=\lim\limits_{x\to 0}\dfrac{\ln\left(\frac{\cos x+1}{2}\right)}{x^2}$

$=\lim\limits_{x\to 0}\dfrac{\cos x-1}{2x^2}=-\dfrac{1}{4}$

例 7.

$$\lim_{x\to 0}\frac{[\sin x-\sin(\sin x)]\sin x}{x^4}$$

解: 原式 $=\lim_{x\to 0}\frac{[\sin x-\sin(\sin x)]\sin x}{\sin^4 x}$

$\overset{\sin x=t}{=\!=}\lim_{t\to 0}\frac{t-\sin t}{t^3}=\frac{1}{6}$

例 8 $\lim_{x\to 0}\frac{x^2\cos^2 x-\sin^2 x}{x^4}$

解: $\lim_{x\to 0}\frac{(x\cos x+\sin x)(x\cos x-\sin x)}{x^4}$

$=\lim_{x\to 0}(\cos x+\frac{\sin x}{x})\cdot\lim_{x\to 0}\frac{x\cos x-\sin x}{x^3}$

$=2\lim_{x\to 0}\frac{\cos x-x\sin x-\cos x}{3x^2}=-\frac{2}{3}$

Case 2 1^{∞}

① 凑 $(1+\Delta)^{\frac{1}{\Delta}}$ $(\Delta \to 0)$

② 恒等变形

1. $\lim\limits_{x\to 0} (1-x\sin x)^{\frac{1}{x-\ln(1+x)}}$

解: $\lim\limits_{x\to 0} \left\{ [1+(-x\sin x)]^{\frac{1}{-x\sin x}} \right\}^{\frac{-x\sin x}{x-\ln(1+x)}}$

$= \lim\limits_{x\to 0} e^{\frac{-x\sin x}{x-\ln(1+x)}} = e^{\lim\limits_{x\to 0} \frac{-x^2}{\frac{1}{2}x^2}} = e^{-2}$

2. $\lim\limits_{x\to\infty} (\cos\frac{1}{x})^{x^2}$

解: $\lim\limits_{x\to\infty} \left\{ [1+(\cos\frac{1}{x}-1)]^{\frac{1}{\cos\frac{1}{x}-1}} \right\}^{x^2(\cos\frac{1}{x}-1)}$

$= \lim\limits_{x\to\infty} e^{x^2(\cos\frac{1}{x}-1)}$

$= e^{\lim\limits_{x\to\infty} \frac{\cos\frac{1}{x}-1}{\frac{1}{x^2}}} = e^{-\frac{1}{2}}$

3. $\lim_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}}$

解: 原 $=\lim_{x\to 0}\left\{\left[1+\left(\frac{\arcsin x}{x}-1\right)\right]^{\frac{1}{\frac{\arcsin x}{x}-1}}\right\}^{\frac{1}{x^2}\left(\frac{\arcsin x}{x}-1\right)}$

$=e^{\lim_{x\to 0}\frac{\arcsin x-x}{x^3}}=e^{\frac{1}{6}}$

4. $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{1}{x^3}}$

解: $\lim_{x\to 0}\left\{\left(\frac{\tan x-\sin x}{1+\sin x}+1\right)^{\frac{1+\sin x}{\tan x-\sin x}}\right\}^{\frac{\tan x-\sin x}{(1+\sin x)x^3}}$

$=\lim_{x\to 0} e^{\frac{\tan x-\sin x}{x^3(1+\sin x)}}$

$=e^{\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}}=e^{\frac{1}{2}}$

Case 3 $\frac{\infty}{\infty}$

① 洛必达

记 $\lim\limits_{x\to\infty}\frac{\ln^a x}{x^b}=0 \quad (a>0,\ b>0)$

$\lim\limits_{x\to\infty}\frac{x^a}{b^x}=0 \quad (a>0,\ b>1)$

$\lim\limits_{x\to 0}\frac{a_m x^m+\cdots}{b_n x^n+\cdots}=\begin{cases}\frac{a_m}{b_n} & m=n\\ \infty & m>n\\ 0 & m<n\end{cases}$

例 1.

$\lim\limits_{x\to\infty}\frac{x^8+3x+1}{(x+1)^a-(x-1)^a}=b$. 求 a,b

解: $\lim\limits_{x\to\infty}\frac{x^8+3x+1}{2ax^{a-1}+\cdots}=b$

$\therefore\begin{cases}a-1=8\\ \frac{1}{2a}=b\end{cases}\Rightarrow\begin{cases}a=9\\ b=\frac{1}{18}\end{cases}$

例 2

$\lim\limits_{x\to\infty}\left(\frac{2x^2+3x+4}{x-1}-ax-b\right)=0$, 求 a,b

解: $\lim\limits_{x\to\infty}\frac{2x^2+3x+4-(ax+b)(x-1)}{x-1}=0$

$\therefore\begin{cases}a=2\\ b-a=-3\end{cases}\Rightarrow\begin{cases}a=2\\ b=-1\end{cases}$

例 3 $\lim\limits_{x\to\infty} \dfrac{\ln(3x^2+x+4)}{\ln(2x^4-x^2+3)}$

解 $\lim\limits_{x\to\infty} \dfrac{6x+1}{3x^2+x+4} \cdot \dfrac{2x^4-x^2+3}{8x^3-2x}$

$= \lim\limits_{x\to\infty} \dfrac{12x^5+\cdots}{24x^5+\cdots} = \dfrac{1}{2}$

对于 $0\cdot\infty \Rightarrow \begin{cases} \dfrac{0}{0} \\ \dfrac{\infty}{\infty} \end{cases}$

Case 4 $\infty-\infty$ 型 { 无分母 / 有分母

1. $\lim\limits_{x\to+\infty}\left[x-x^2\ln\left(1+\frac{1}{x}\right)\right]$

解: 原 $=\lim\limits_{x\to+\infty} x^2\left[\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)\right]$

$=\lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}-\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}$

$\xrightarrow{\frac{1}{x}=t}\lim\limits_{t\to0}\dfrac{t-\ln(1+t)}{t^2}=\frac{1}{2}$

2. $\lim\limits_{x\to+\infty}\left[(x+1)\arctan x-\frac{\pi}{2}x\right]$

解: 原 $=\lim\limits_{x\to+\infty}\left[x\arctan x-\frac{\pi}{2}x\right)+\lim\limits_{x\to+\infty}\arctan x$

$=\frac{\pi}{2}+\lim\limits_{x\to+\infty}\dfrac{\arctan x-\frac{\pi}{2}}{\frac{1}{x}}$

$=\frac{\pi}{2}+\lim\limits_{x\to+\infty}\dfrac{\frac{1}{1+x^2}}{-\frac{1}{x^2}}=\frac{\pi}{2}-1$

3. $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+2x+3\sin x}-x\right)$

解: 原 $=\lim\limits_{x\to+\infty}\dfrac{2x+3\sin x}{\sqrt{x^2+2x+3\sin x}+x}$

$=\lim\limits_{x\to+\infty}\dfrac{2+3\frac{\sin x}{x}}{\sqrt{1+\frac{2}{x}+3\frac{\sin x}{x^2}}+1}$

$=\frac{2}{2}=1$

有分母

1. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\operatorname{arctan}^2 x}\right)$

解: 原 $=\lim_{x\to 0}\frac{(\operatorname{arctan}x+x)(\operatorname{arctan}x-x)}{x^4}$

$=\lim_{x\to 0}\frac{\operatorname{arctan}x+x}{x}\cdot\frac{\operatorname{arctan}x-x}{x^3}$

$=\lim_{x\to 0}\ 2\cdot\left(-\frac{1}{3}\right)=-\frac{2}{3}$

$\left.\begin{matrix}\infty^0\\ 0^0\end{matrix}\right\}\Rightarrow e^{\ln(\cdots)}$

1. $\lim_{x\to 0^+} x^{\sin x}$

解: $\lim_{x\to 0^+} e^{\sin x\ln x}=e^{\lim_{x\to 0^+}\sin x\ln x}=e^{\lim_{x\to 0^+} x\ln x}$

$=\lim_{x\to 0^+} e^{\frac{\ln x}{\frac{1}{x}}}=e^{2\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}}=e^{-\lim_{x\to 0^+} x}=e^0=1$

Part Ⅱ 连续与间断.

— defs.

1. 连续

① $f(x)$ 在 $x=a$ 连续：

If $\lim\limits_{x\to a} f(x)=f(a)$，即 $f(a-0)=f(a+0)=f(a)$

then $f(x)$ 在 $x=a$ 连续

② $f(x)$ 在 $[a,b]$ 上连续

If $\begin{cases} f(x) \text{ 在 } (a,b) \text{ 内处处连续} \\ f(a+0)=f(a) \quad f(b-0)=f(b) \end{cases}$

记 $f(x)\in C[a,b]$

2. 间断 If $\exists x_0$ $f(x)\neq f(a)$ 间断

间断点分类

① 第一类：$f(a-0)$, $f(a+0)$ $\exists$ $\begin{cases} f(a-0)=f(a+0) \text{ 且 } \neq f(a) & x=a \text{ 可去} \\ f(a-0)\neq f(a+0) & x=a \text{ 跳跃} \end{cases}$

② 第二类：$f(a-0)$, $f(a+0)$ 至少一个不 $\exists$.

型四 间断点及分类

1. $f(x)=\dfrac{\ln|x|}{x^2-1}e^{\frac{1}{x-2}}$

解：1° $x=-1,0,1,2$ 为间断点，

2° $\lim\limits_{x\to-1}f(x)=\lim\limits_{x\to-1}\dfrac{1}{x-1}e^{\frac{1}{x-2}}\cdot\dfrac{\ln|x|}{x+1}$

$=-\dfrac{e^{-\frac{1}{3}}}{2}\lim\limits_{x\to-1}\dfrac{\ln(-x)}{x+1}$

$=-\dfrac{1}{2}e^{-\frac{1}{3}}\lim\limits_{x\to-1}\dfrac{\ln[1-(x+1)]}{x+1}=-\dfrac{1}{2}e^{-\frac{1}{3}}$

$x=-1$ 为可去间断点

3° $\lim\limits_{x\to0}f(x)=+\infty$，$x=0$，第二类

4° $\lim\limits_{x\to1}f(x)=\lim\limits_{x\to1}\dfrac{1}{x+1}e^{\frac{1}{x-2}}\cdot\dfrac{\ln x}{x-1}$

$=\dfrac{1}{2}e^{-1}\lim\limits_{x\to1}\dfrac{\ln[1+(x-1)]}{x-1}=\dfrac{1}{2e}$

$x=1$ 为可去

5° $\lim\limits_{x\to2^-}f(x)=\dfrac{\ln2}{5}\lim\limits_{x\to2^-}e^{\frac{1}{x-2}}=0$

$\lim\limits_{x\to2^+}f(x)=\dfrac{\ln2}{5}(+\infty)=+\infty$

$x=2$ 为第二类

2. $f(x)=\frac{1-e^{\frac{1}{x-1}}}{1+e^{\frac{1}{x-1}}}$ 间断点分类

解: 1° $x=1$ 为间断点,

2° $\lim\limits_{x\to 1^-} f(x)=1$ $\lim\limits_{x\to 1^+} f(x)=-1$

$\therefore x=1$ 跳跃.

二、闭区间上连续函数性质

1. 最值: 设 $f(x)\in C[a,b] \Rightarrow \exists m, M$ $\forall x$ $m\le f(x)\le M$

2. 有界 $\Rightarrow \exists k$ $|f(x)|\le k$.

3. 零点 $f(x)\in C[a,b]$, $f(a)f(b)<0 \Rightarrow \exists c\in[a,b]$, $f(c)=0$

介值: $f(x)\in C[a,b] \Rightarrow m, M$, $[m,M]$ 值域

$\forall \mu\in[m,M]$ 称为介值

4. 介值定理 $f(x)\in C[a,b]$, $\forall \mu\in[m,M]$, $\exists \xi\in[a,b]$, $f(\xi)=\mu$.

Notes. $f(x)\in C[a,b]$

① 函数之和

② $\exists \xi\in[a,b]$

$\Rightarrow$ 介值定理.

1. $f(x) \in C[0,2]$，$f(0)+2f(1)+3f(2)=12$，证 $\exists c \in [0,2]$，$f(c)=2$

解：$f(x) \in C[0,2] \Rightarrow \exists\ m, M$

$\therefore 6m \le f(0)+2f(1)+3f(2) \le 6M \Rightarrow m \le 2 \le M$

$\exists c \in [0,2]$，$f(c)=2$

2. $f(x) \in C[a,b]$ $p>0$，$q>0$，证

$\exists \xi \in [a,b]$，使 $pf(a)+qf(b)=(p+q)f(\xi)$

解：$f(x) \in C[a,b] \Rightarrow \exists\ m, M$

$(p+q)m \le pf(a)+qf(b) \le (p+q)M \Rightarrow m \le \frac{pf(a)+qf(b)}{p+q} \le M$

$\exists \xi \in [a,b]$，$f(\xi)=\frac{pf(a)+qf(b)}{p+q}$

即 $(p+q)f(\xi)=pf(a)+qf(b)$.

二、导数与微分

一. 基本概念

1. 导数 — $y=f(x)$ $(x\in D)$，$x_0\in D$，$x_0+\Delta x\in D$

$$\Delta y = f(x_0+\Delta x) - f(x_0)$$

If $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ $\exists$ 称 $f(x)$ 在 $x=x_0$ 可导.

$\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ 为 $f(x)$ 在 $x=x_0$ 导数 $f'(x_0)$ $\frac{dy}{dx}\Big|_{x=x_0}$

Notes.

① $f'(x_0)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$

$f'(x_0)=\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

② $\Delta x\to 0$ $\begin{cases}\Delta x\to 0^+\\ \Delta x\to 0^-\end{cases}$ 或 $x\to a$ $\begin{cases}x\to a^-\\ x\to a^+\end{cases}$

$$\lim_{\Delta x\to 0^+}\frac{\Delta y}{\Delta x}=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\triangleq f'_+(x_0)$$

$$\lim_{\Delta x\to 0^-}\frac{\Delta y}{\Delta x}=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\triangleq f'_-(x_0)$$

$$f'(a)\ \exists \iff f'_+(x_0)=f'_-(x_0)$$

1. $f(x)=|\ln x|$，求 $f'(1)$？

解：$f'_-(1)=\lim\limits_{x\to1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to1^-}\dfrac{-\ln x}{x-1}=-1$

$f'_+(1)=\lim\limits_{x\to1^+}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to1^+}\dfrac{\ln x}{x-1}=1$

$\because f'_-(1)\neq f'_+(1)$　$\therefore f'(1)$ 不存在.

2. $$f(x)=\begin{cases}\dfrac{x\cdot 2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}, & x\neq0\\ 0, & x=0\end{cases}$$ 求 $f'(0)$？

解：$\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$

$\therefore f'_-(0)=0$　$f'_+(0)=1$　$\therefore f'(0)$ 不存在

③ $f(x)$ 在 $x=a$ 可导 $\Rightarrow$ $f(x)$ 在 $x=a$ 连续

$f'(a)\exists \Rightarrow \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}\exists$

证 "$\Rightarrow$" $\lim\limits_{x\to a}[f(x)-f(a)]=0$

$\Rightarrow \lim\limits_{x\to a}f(x)=f(a)$ 即 $f(x)$ 在 $x=a$ 连续

证 "$\not\Leftarrow$" 反例 $f(x)=2x+|x|$

$f'_-(0)=1$, $f'_+(0)=3$ $\Rightarrow f'(0)$ 不存在.

④ $f(x)$ 连续. 若 $\lim\limits_{x\to a}\dfrac{f(x)-b}{x-a}=A$ $\Longrightarrow$ $\begin{cases} f(a)=b \\ f'(a)=A \end{cases}$

2. 可微 $y=f(x)$ $(x\in D)$. $x_0\in D$, $\Delta y=f(x_0+\Delta x)-f(x_0)$

If $\Delta y=A\Delta x+o(\Delta x)$, 称 $f(x)$ 在 $x=x_0$ 处可微.

其中 $A\Delta x$ 称为 $x=x_0$ 处微分, 记 $dy\big|_{x=x_0}=A\,dx$

Notes

① $f(x)$ 在 $x=x_0$ 可导 $\Leftrightarrow$ 可微

证: "$\Rightarrow$" $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x_0)$

$\Rightarrow \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(x_0)+\alpha \quad \lim\limits_{\Delta x\to 0}\alpha=0$

$\Rightarrow \Delta y=f'(x_0)\Delta x+\alpha(\Delta x)$

证 "$\Leftarrow$" $\Delta y=A\Delta x+o(\Delta x)$

$\Rightarrow \frac{\Delta y}{\Delta x}=A+\frac{o(\Delta x)}{\Delta x}$

$\Rightarrow \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}A+\frac{o(\Delta x)}{\Delta x}=A$ 且 $f'(x_0)=A$

① If $\Delta y=A\Delta x+o(\Delta x)$，则 $A=f'(x_0)$

② $y=f(x)$ 处 n 可导 $dy=df(x)=f'(x)dx$ (不定积分)

二. 求导工具

Notes

① 基本初函 $\begin{cases} x^{\alpha} \\ a^{x} \quad (a>0 \text{ 且 } a\neq 1) \\ \log_a x \quad (a>0 \text{ 且 } a\neq 1) \\ \sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x \\ \arctan x,\ \operatorname{arccsin} x,\ \arccos x,\ \operatorname{arccot} x \end{cases}$

② 初等函数 由 $\begin{cases} \text{常数} \\ \text{基本初函} \end{cases}$ $\xrightarrow[\text{复合}]{\text{四则}}$ 初函

1. $f(x)=x(x-1)(x+2)\cdots(x-99)(x+100)$，求 $f'(0)$

解：$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}(x-1)(x+2)\cdots(x-99)(x+100)$

$=100!$

另法：

$f(x)=(x-1)(x-2)\cdots(x-99)(x+100)+x\varphi(x)$

$f'(0)=(x-1)(x+2)\cdots(x+100)=100!$

复合

$y=f(u)$ 可导，$u=\varphi(x)$ 可导，且 $\varphi'(x)\neq 0$，则 $y=f[\varphi(x)]$ 可导

且 $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=f'(u)\cdot\varphi'(x)=f'[\varphi(x)]\varphi'(x)$

证：$f'(u)=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta u}$

$\varphi'(x)=\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\neq 0 \Rightarrow \Delta u=0(\Delta x)$

$\therefore \frac{dy}{dx}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$

$=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$

(四) 反函数求导

↳ 函数严格单调才会有反函数 （1对1）

$y=f(x) \longrightarrow x=\varphi(y)$

1. 求 $y=\ln(x+\sqrt{1+x^2})$ 的反函数

解：$x+\sqrt{1+x^2}=e^y$

同 $(-x+\sqrt{1+x^2})(x+\sqrt{1+x^2})=e^{-y}$

$\therefore -x+\sqrt{1+x^2}=e^{-y}$

$\therefore x=\frac{e^y-e^{-y}}{2}$

即 $x=\frac{e^y-e^{-y}}{2}$

Th. $y=f(x)$ 可导且 $f'(x)\neq 0 \Rightarrow x=\varphi(y)$ 可导，且 $\varphi'(y)=\frac{1}{f'(x)}$

证：$f'(x)\neq 0 \Rightarrow y=f(x)$ 严格单调 $\Rightarrow \exists\, x=\varphi(y)$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\neq 0 \Rightarrow \Delta y=0(\Delta x)$$

$$\varphi'(y)=\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y}=\lim_{\Delta x\to 0}\frac{1}{\frac{\Delta y}{\Delta x}}=\frac{1}{f'(x)}$$

型一 定义求导

1. $f(x)=\begin{cases}\ln(1-2x) & x<0\\ \sin ax & x\geq 0\end{cases}$ 已知 $f'(0)$ 存在，求 a

解：$f(0-0)=f(0+0)=f(0)$ $\Rightarrow$ 连续

$$\left.\begin{aligned}f'_-(0)&=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x}=-2\\ f'_+(0)&=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x}=a\end{aligned}\right\}\Rightarrow a=2$$

2. $f(x)$ 连续 $\lim\limits_{x\to 1}\frac{f(x)-1}{x-1}=2$，求 $\lim\limits_{x\to 1}\frac{f(x)-1}{x^2-1}$

解：$\lim\limits_{x\to 1}\frac{f(x)-1}{x-1}=2\Rightarrow\begin{cases}f(1)=1\\ f'(1)=2\end{cases}$

$$\text{原}=\lim_{x\to 1}\frac{f(x)-1}{x+1}\cdot\frac{f(x)-1}{x-1}=\lim_{x\to 1}\frac{f(x)-1}{x+1}=2$$

型二　显函数求导.

- 公式
- 四则
- 复合.

型三　隐函数求导.

1. $e^{xy} = x^2+y+1$　求 $\frac{dx}{dy}$

解: $e^{xy}(y+xf') = 2x+f'$

$\Rightarrow f' = \frac{ye^{xy}-2x}{-1-xe^{xy}}$

2. $e^{xy} + \tan(xy) = y$. 求 $\frac{dy}{dx}|_{x=0}$

解: ① $x=0 \Rightarrow y=1$

② $e^{xy}(y+xf') + \sec^2(xy)(y+xf') = f'$

$\Rightarrow f'|_{x=0} = 2$

型四 参数方程确定的函数导数

$\varphi(t)$，$\psi(t)$ 可导且 $\varphi'(t)\neq 0$，由 $\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$

$\Rightarrow y=f(x)$ 且 $\dfrac{dy}{dx}=\dfrac{\psi'(t)}{\varphi'(t)}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$

1. 设 $\begin{cases} x=t-\ln(1+t) \\ y=t^2 \end{cases}$ 求 $\dfrac{dy}{dx}$，$\dfrac{d^2y}{dx^2}$

解：$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}\Rightarrow \dfrac{dy}{dt}=2(t+1)$

$\dfrac{d^2y}{dx^2}=\dfrac{d\left(\frac{dy}{dx}\right)}{dx}=\dfrac{d\left(\frac{dy}{dx}\right)}{dt}\Big/\dfrac{dx}{dt}=\dfrac{2}{\frac{t}{1+t}}=\dfrac{2(1+t)}{t}$

2. $\begin{cases} x=\arctan t \\ y=\ln(1+t^2) \end{cases}$ 求 $\dfrac{dy}{dx}$，$\dfrac{d^2y}{dx^2}$

解：$\dfrac{dy}{dt}=\dfrac{2t}{1+t^2}$ $\dfrac{dx}{dt}=\dfrac{1}{1+t^2}$ $\therefore \dfrac{dy}{dx}=2t$

$\dfrac{d^2y}{dx^2}=\dfrac{2}{\frac{2t}{1+t^2}}=2(1+t^2)$

型五. 分段函数求导.

1. $f(x)=\begin{cases} e^{2x} & x<0 \\ ax+b & x\geq 0 \end{cases}$ $f'(0)$ 存, 求 a,b

$\therefore \lim\limits_{x\to 0^-} e^{2x} = \lim\limits_{x\to 0^+} ax+b = b = 1$

$\therefore b=1$ $\Rightarrow \begin{cases} a=2 \\ b=1 \end{cases}$

$f'_-(0) = \lim\limits_{x\to 0^-} \frac{e^{2x}-1}{x} = 2$

$f'_+(0) = \lim\limits_{x\to 0^+} \frac{ax+b-1}{x} = a$ $\Rightarrow a=2$

2. $f(x)=\begin{cases} \frac{\sin x}{x} & x\neq 0 \\ 1 & x=0 \end{cases}$ 求 $f'(x)$

解

① $x\neq 0$ 时, $f'(x) = \frac{x\cos x-\sin x}{x^2}$

② $x=0$ 时, $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = \lim\limits_{x\to 0} \frac{\sin x-x}{x^2} = 0$

$\therefore f'(x)=\begin{cases} \frac{x\cos x-\sin x}{x^2} & x\neq 0 \\ 0 & x=0 \end{cases}$

$f'(0) = \lim\limits_{x\to 0} \frac{x\cos x-\sin x}{x^2} = \frac{1}{2}\lim\limits_{x\to 0} \sin x = 0 = f'(0)$

$\therefore f'(x)$ 在 $x=0$ 连续

型六 高阶导数

法一. 归纳法

1. $y=\frac{1}{2x+3}$，求 $y^{(n)}$

解：$y=(2x+3)^{-1}$

$y'=(-1)\cdot 2(2x+3)^{-2}$

$y''=(-1)(-2)\cdot 2^2(2x+3)^{-3}$

$\vdots$

$y^{(n)}=(-1)^n\cdot n!\cdot 2^n(2x+3)^{-(n+1)}$

记 $\left[\frac{1}{ax+b}\right]^{(n)}=\frac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}$

2. $y=\frac{1}{x^2-x-2}$，求 $y^{(n)}$

解：$y=\frac{1}{3}\left(\frac{1}{x-2}-\frac{1}{x+1}\right)$

$\therefore y^{(n)}=\frac{1}{3}\left[\frac{(-1)^n\cdot n!}{(x-2)^{n+1}}-\frac{(-1)^n n!}{(x+1)^{n+1}}\right]$

3. $y = e^x \sin x$ ，求 $y^{(n)}$

解：$y' = e^x \sin x + e^x \cos x$

$= e^x \sin\left(x + \frac{\pi}{4}\right) \cdot \sqrt{2}$

$y'' = (\sqrt{2})^2 e^x \sin\left(x + \frac{\pi}{2}\right)$

$\vdots$

$y^{(n)} = (\sqrt{2})^n e^x \sin\left(x + \frac{n}{4}\pi\right)$

第三章
中值定理与一元微分学应用

预备知识

1. 极值点：$y=f(x)$ $(x\in D)$，$x_0\in D$

① 若 $\exists\ \delta>0$，$0<|x-x_0|<\delta$ 时，使得 $f(x)<f(x_0)$

x_0 为极大点。

② 若 $\exists\ \delta>0$，$0<|x-x_0|<\delta$ 时，使得 $f(x)>f(x_0)$

x_0 为极小点。

2.

$$f'(x_0)\begin{cases}>0\\<0\\=0\\\text{不}\exists\end{cases}$$

① $f'(x_0)>0$ ∴ $f'(x_0)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}>0$

$\exists\ \delta>0$，当 $0<|x-a|<\delta$，$\dfrac{f(x)-f(a)}{x-a}>0$

$\begin{cases}f(x)<f(a) & x\in(a-\delta,a) \text{ ：左小}\\ f(x)>f(a) & x\in(a,a+\delta) \text{ ：右大}\end{cases}$

$f'(x_0)>0\Rightarrow$ 左小右大 $\Downarrow$ 非极值点。

② $f'(a)<0$ ∵ $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}<0$，$\exists\delta>0$ 当 $0<|x-a|<\delta$ 使 $\frac{f(x)-f(a)}{x-a}<0$

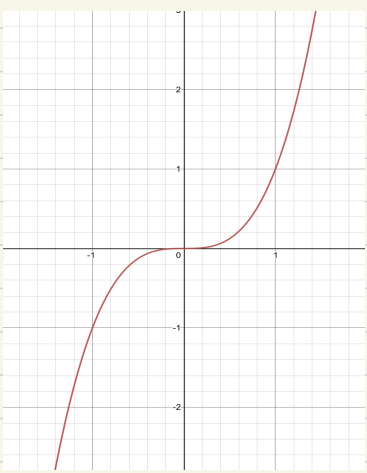
∴ $\begin{cases} f(x)>f(a) & x\in(a-\delta,a) \\ f(x)<f(a) & x\in(a,a-\delta) \end{cases}$

$f'(a)<0 \Rightarrow$ 左大右小 $\Rightarrow$ 非极值点.

结论：

① $f(x)$ 在 $x=a$ 取极值 $\Rightarrow f'(a)=0$ 或 $f'(a)$ 不∃

② $f(x)$ 可导 且在 $x=a$ 取极值 $\Rightarrow f'(a)=0$



反例.

$f(x)=x^3$

② $f'_+(a)>0 \Rightarrow$ 右大

即 $\exists x_1>a,\ f(x_1)>f(a)$

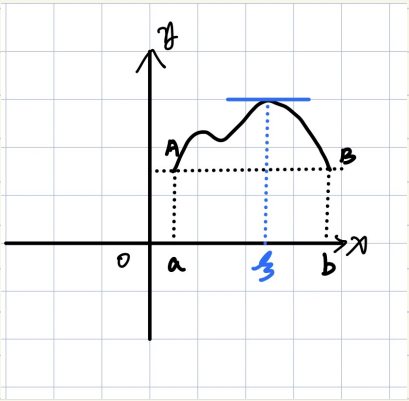
$f'_+(a)<0 \Rightarrow$ 右小

即 $\exists x_0<a,\ f(x_1)<f(a)$

Part 1. 中值定理

Th1. 罗尔定理

If $\begin{cases} ① f(x)\in C[a,b] \\ ② (a,b) \text{ 内可导} \\ ③ f(a)=f(b) \end{cases}$



则 $\exists \xi\in(a,b)$.

$f(\xi)=0$

证：$f(x) \in C[a,b] \Rightarrow \exists\, m, M$

① $m = M$　　$f(x) \equiv C_0$　　$\forall \xi \in (a,b)$，$f'(\xi) = 0$

② $m < M$

$\because f(a) = f(b)$，$\therefore$ m、M 至少有一个在 (a,b) 内取

设 $\exists \xi \in (a,b)$，使 $f(\xi) = M \Rightarrow f'(\xi) = 0$ 或 $f'(\xi)$ 不存在

$\because f(x)$ 在 (a,b) 内可导 $\Rightarrow f'(\xi) = 0$

例：$f(x) \in C[0,2]$，$(0,2)$ 内可导，且 $f(0) = 1/2$，$f(1) = 0$

$f(2) = 2$，证：$\exists \xi \in (0,2)$ $f'(\xi) = 0$

解：1° $\varphi(x) = f(x) - \frac{1}{2}$

$\varphi(1) = -\frac{1}{2}$　　$\varphi(2) = \frac{3}{2}$

$\because \varphi(1) \cdot \varphi(2) < 0$　$\therefore \exists\, C \in (1,2)$ 使 $\varphi(C) = 0$

$\therefore f(C) = \frac{1}{2} = f(0)$

2° Rolle 定理：$\exists \xi \in (0, C)$，$f'(\xi) = 0$

例 2. $f(x) \in C[0,2]$, $(0,2)$ 内可导, $f(0)=1$, $f(1)+2f(2)=3$

证 $\exists \xi \in (0,2)$, $f'(\xi)=0$

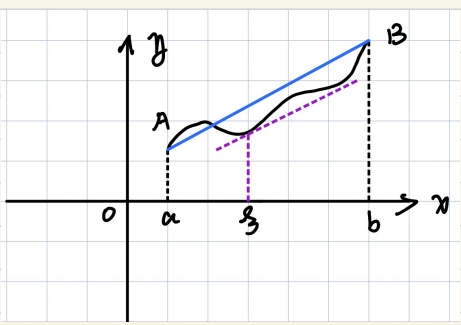
解: 证: $f(x) \in C[1,2] \Rightarrow \exists\, m, M$

$3m \le f(1)+2f(2) \le 3M \Rightarrow m \le 1 \le M$

$\therefore \exists\, c_0 \in [1,2]$, $f(c_0)=1$.

$f(0)=f(c_0)$ $\therefore \exists \xi \in (0,c_0) \subset (0,2)$, $f'(\xi)=0$

Th 2. (Lagrange) If $\begin{cases} f(x) \in C[a,b] \\ f(x) \text{ 在 } (a,b) \text{ 内可导} \end{cases} \Rightarrow \exists \xi \in (a,b)$, $f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$



证:

分析: L: $y=f(x)$

L_{AB}: $y-f(a)=\dfrac{f(b)-f(a)}{b-a}(x-a)$

令 $\varphi(x)$ = 曲 − 直

$=f(x)-f(a)-\dfrac{f(b)-f(a)}{b-a}(x-a)$

$\varphi(x) \in C[a,b]$, $\varphi(x)$ 在 (a,b) 内可导 $\varphi(a)=0$, $\varphi(b)=0$

Rolle $\Rightarrow \exists \xi \in (a,b)$, $\varphi'(\xi)=0$ 即 $\varphi'(\xi)=f'(\xi)-\dfrac{f(b)-f(a)}{b-a}=0$

即 $\exists \xi \in (a,b)$, 使 $f'(\xi)=\dfrac{f(b)-f(a)}{b-a}$

Notes.

① If $f(a)=f(b)$ 则 $L \Rightarrow R$

② $f'(\xi)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow f'(\xi)(b-a)=f(b)-f(a)$

$\Updownarrow$

$f(b)-f(a)=f'[a+(b-a)\theta](b-a) \quad (0<\theta<1)$

Th3 Cauchy

If $\begin{cases} ① f(x), g(x) \in C[a,b] \\ ② f(x), g(x) \text{ } (a,b) \text{ 内可导} \\ ③ g'(x) \neq 0 \quad (a<x<b) \end{cases}$

则 $\exists \xi \in (a,b)$ 使 $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$

分析

① $g'(x) \neq 0 \quad (a<x<b) \Rightarrow \begin{cases} g'(\xi) \neq 0 \\ g(b)-g(a) \neq 0 \end{cases}$

② $L: \varphi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$

$C: \varphi(x)=f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]$

证：

$\because \varphi(a)=0,\ \varphi(b)=0$

Rolle $\Rightarrow \exists \xi \in (a,b),\ \varphi'(\xi)=0$

$$\Rightarrow f'(\xi)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(\xi)=0$$

$$\Rightarrow \frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Notes

If $g(x)=f(x)$ $\quad C \Rightarrow L$

型一 证明 $f^{(n)}(\xi)=0$

Case 1. $f'(\xi)=0$: $f(c_1)=f(c_2)$

Case 2. $f''(\xi)=0$: $\begin{cases} f(c_1)\ f(c_2)\ f(c_3) \\ f'(c_1)=f'(c_2) \end{cases}$

例 1.

例1 设 $f(x)\in C[0,2]$，在$(0,2)$内可导，且$3f(0)=f(1)+2f(2)$，证明：存在$\xi\in(0,2)$，使得 $f'(\xi)=0$.

证：

1° $f(x)\in C[1,2]$ $\exists\ m,M$

$\therefore\ 3m\le f(1)+2f(2)\le 3M \Rightarrow m\le f(c)\le M$

$\therefore \exists c\in[1,2]$，$f(c)=f(0)$

2° $\exists\ \xi\in(0,c)\subset(0,2)$，$f'(\xi)=0$

例 2.

例4 设 $f(x)$ 二阶可导，且$\lim\limits_{x\to 0}\dfrac{f(x)-1}{x}=0$，又 $f(1)=1$，证明：存在 $\xi\in(0,1)$，使得 $f''(\xi)=0$.

证：

1° $\lim\limits_{x\to 0}\dfrac{f(x)-1}{x}=0 \Rightarrow \begin{cases} f(0)=1 \\ f'(0)=0 \end{cases}$；

2° $\because f(0)=f(1)$

$\therefore\ \exists c\in(0,1)$，$f'(c)=0$，$f'(0)=f'(c)$ $\therefore f''(\xi)\in(0,c)\subset(0,1)$

使 $f''(\xi)=0$

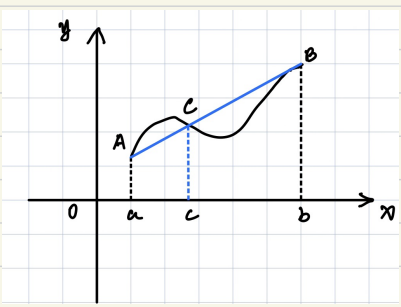
例3. $f(x)$ 二阶可导，$\lim\limits_{x\to 0}\frac{f(x)}{x}=1$　$f(1)=1$　证 $\exists \xi\in(0,1)$，$f''(\xi)=0$

证：1° $\lim\limits_{x\to 0}\frac{f(x)}{x}=1 \Rightarrow \begin{cases} f(0)=0 \\ f'(0)=1 \end{cases}$

2° $\exists \xi_1\in(0,1)$，$f'(\xi_1)=\frac{f(1)-f(0)}{1-0}=1$

3° $f'(\xi_1)=f'(0)=1$　$\therefore \exists \xi\in(0,\xi_1)$，$f''(\xi)=0$

例4. $f(x)$ 二阶可导．$L: y=f(x)$，连接 $A(a,f(a))$，$B(b,f(b))$ 及 $C(c,f(c))$
其中 $a<c<b$，证 $\exists \xi\in(a,b)$，$f''(\xi)=0$



证：1° $\exists \xi_1\in(a,c)$　$\xi_2\in(c,b)$

使 $f'(\xi_1)=\frac{f(c)-f(a)}{c-a}$

$f'(\xi_2)=\frac{f(b)-f(c)}{b-c}$

2° $f'(\xi_1)=f'(\xi_2)$

$\exists \xi\in(\xi_1,\xi_2)$，$f''(\xi)=0$

型二 仅有ξ

① 还原法 {仅有ξ / 导数差一阶 / 两项}

工具 $\frac{f'}{f}=(\ln f)'$, $\frac{f''}{f'}=(\ln f')'$

例:

例1 设 $f(x)\in C[0,1]$,在$(0,1)$内可导,$f(1)=0$,证明:存在$\xi\in(0,1)$,使得 $2f(\xi)+\xi f'(\xi)=0$.

解:

$$2f(x)+xf'(x)=0 \Rightarrow \frac{f'(x)}{f(x)}+\frac{2}{x}=0$$

$$\Rightarrow (\ln f)'+(2\ln x)'=0$$

$$\Rightarrow [\ln x^2 f(x)]'=0$$

证 $\varphi(x)=x^2f(x)$ $\because \varphi(0)=0$, $\varphi(1)=0$

$\therefore \exists \xi\in(0,1)$, $\varphi'(\xi)=2\xi f(\xi)+\xi^2 f'(\xi)=0$

而 $\xi\neq 0 \Rightarrow 2f(\xi)+\xi f'(\xi)=0$

例2.

例2 设 $f(x)\in C[a,b]$，在 (a,b) 内可导，$f(a)=f(b)=0$，证明：存在 $\xi\in(a,b)$，使得 $f'(\xi)-2f(\xi)=0$.

证： $f'(x)-2f(x)=0 \Rightarrow \dfrac{f'(x)}{f(x)}-2=0$

$\Rightarrow [\ln f(x)]'-(2x)'=0$

$\Rightarrow \{\ln[f(x)\cdot e^{-2x}]\}'=0$

设 $\varphi(x)=f(x)\cdot e^{-2x}$ $\therefore$ $\varphi(a)=0=\varphi(b)$

$\therefore \exists \xi\in(a,b)$，$\varphi'(\xi)=f'(\xi)e^{-2\xi}-2e^{-2\xi}f(\xi)=0$

$\because e^{-2\xi}\neq 0$ $\therefore$ $f'(\xi)-2f(\xi)=0$

例3

例4 设 $f(x)$ 在 $[0,1]$ 上二阶可导，且 $f(0)=f(1)$，证明：存在 $\xi\in(0,1)$，使得

$$f''(\xi)=\frac{2f'(\xi)}{1-\xi}.$$

分析

证： $\dfrac{f''(x)}{f'(x)}-\dfrac{2}{1-x}=0$

$[\ln f'(x)]'+[2\ln(x-1)]'=0 \Rightarrow \ln[f'(x)\cdot(x-1)^2]=0$

$\therefore$ 令 $\varphi(x)=f'(x)(x-1)^2$ $\varphi(1)=0$ $f(0)=f(1)$ $\therefore \exists c\in(0,1)$，$f(c)=0$

$\therefore \varphi(c)=\varphi(1)$ $\therefore \exists \xi\in(c,1)$，$\varphi'(\xi)=0$

$\therefore \varphi'(\xi)=f''(\xi)(\xi-1)^2+2(\xi-1)f'(\xi)=0$

$\Rightarrow f''(\xi)(\xi-1)+2f'(\xi)=0 \Rightarrow f''(\xi)=\dfrac{2f'(\xi)}{1-\xi}$

② 分组法.

例1: $f(x)$ 二阶可导. $\lim_{x\to 0}\frac{f(x)}{x}=1$. $f(1)=1$, 证 $\exists \xi\in(0,1)$, $f''(\xi)-f'(\xi)+1=0$

分析: $f''(x)-f'(x)+1=0$

$\Rightarrow \frac{(f'-1)'}{g'}-\frac{(f'-1)}{g}=0 \Rightarrow \frac{g'}{g}-1=0 \Rightarrow (\ln g)'+(\ln e^{-x})'=0$

$\Rightarrow [\ln g e^{-x}]'=0$

证: 令 $\varphi(x)=e^{-x}[f'(x)-1]$

$\lim_{x\to 0}\frac{f(x)}{x}=1 \Rightarrow \begin{cases} f(0)=0 \\ f'(0)=1 \end{cases} \Rightarrow \varphi(0)=0$, $\exists c\in(0,1)$ $f'(c)=\frac{f(1)-f(0)}{1-0}=1$

$\therefore \varphi(c)=0 \quad \therefore \exists \xi\in(0,c)$ $\varphi'(\xi)=0$

$\varphi'(\xi)=-e^{-\xi}[f'(\xi)-1]+f''(\xi)e^{-\xi}=0$

$\Rightarrow f''(\xi)-f'(\xi)+1=0$

型三　有 ξ，有 a, b

Case 1.　ξ 与 a, b 可分离

ξ 与 a, b 分离 $\Rightarrow$ a, b 侧 $\begin{cases} f(b)-f(a) \text{ 或 } \dfrac{f(b)-f(a)}{b-a} \\ \dfrac{f(b)-f(a)}{g(b)-g(a)} \end{cases}$

Case 2　ξ 与 a, b 不可分

$\begin{cases} \xi \to x \\ \text{去分母，移项} \end{cases}$ $\Rightarrow$ 式子 $=0$ $\Rightarrow$ $(\quad)' = 0$ (无 b)

例1.

例2　设 $ab>0(a<b)$，证明：存在 $\xi \in (a,b)$，使得 $ae^b - be^a = (a-b)(1-\xi)e^{\xi}$.

解：

分析：$ae^b - be^a = (a-b)(1-\xi)e^{\xi}$

$\Downarrow$

$$\frac{ae^b - be^a}{a-b} = (1-\xi)e^{\xi}$$

$$\frac{\frac{e^b}{b} - \frac{e^a}{a}}{\frac{1}{b} - \frac{1}{a}}$$

证 令 $f(x)=\frac{e^x}{x}$, $g(x)=\frac{1}{x}$ $g'(x)=-\frac{1}{x^2}\neq 0$

$\exists \xi\in(a,b)$ 使 $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$

$$\frac{\frac{e^b}{b}-\frac{e^a}{a}}{\frac{1}{b}-\frac{1}{a}}=\frac{\frac{\xi e^\xi-e^\xi}{\xi^2}}{-\frac{1}{\xi^2}}=e^\xi-\xi e^\xi=(1-\xi)e^\xi$$

即 $\frac{ae^b-be^a}{a-b}=(1-\xi)e^\xi$

例2.

例1 设 $f(x),g(x)\in C[a,b]$，在 (a,b) 内可导，且 $g'(x)\neq 0$，证明：存在 $\xi\in(a,b)$，使得

$$\frac{f(a)-f(\xi)}{g(\xi)-g(b)}=\frac{f'(\xi)}{g'(\xi)}.$$

分析

解: $f(a)g(x)-f(x)g(x)=f(x)g(x)-g(b)f(x)$

$f(a)g(x)-f(x)g(x)-f(x)g(x)+g(b)f(x)=0$

$\Rightarrow [f(a)g(x)-f(x)g(x)+g(b)f(x)]'=0$

证: 令 $\varphi(x)=f(a)g(x)-f(x)g(x)+f(x)g(b)$

$\varphi(a)=f(a)g(b)$, $\varphi(b)=f(a)g(b)\Rightarrow \exists \xi\in(a,b)$ $\varphi'(\xi)=0$

$\therefore f(a)g'(\xi)-f(\xi)g'(\xi)-f(\xi)g(\xi)+f'(\xi)g(b)=0$

$\Rightarrow \frac{f(a)-f(\xi)}{g(\xi)-g(b)}=\frac{f'(\xi)}{g'(\xi)}$

型四 有多个 η

Case 1 仅有 $f'(\xi)$, $f'(\eta)$ { 找三点 / 两次 L

例1.

例1 设 $f(x)\in C[0,1]$,在$(0,1)$内可导,$f(0)=0$,$f(1)=1$.

(1) 证明:存在 $c\in(0,1)$,使得 $f(c)=1-c$;

(2) 证明:存在 $\xi,\eta\in(0,1)(\xi\neq\eta)$,使得 $f'(\xi)f'(\eta)=1$.

解: (1) 令 $\varphi(x)=f(x)-1+x$

$\varphi(0)=-1$ $\varphi(1)=1$ $\because \varphi(0)\varphi(1)<0$

$\therefore \exists\, c\in(0,1)$, $\varphi(c)=0$ 即 $f(c)=1-c$

(2) $\exists\,\xi\in(0,c)$ $\eta\in(c,1)$

$f'(\xi)=\frac{f(c)-f(0)}{c}=\frac{1-c}{c}$ $f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{c}{1-c}$

$\therefore f'(\xi)\cdot f'(\eta)=1$

例2. 证 $f'(\xi)f'(\eta)=1$.

例2 设 $f(x)$ 在$[0,1]$上连续，在$(0,1)$内可导，且 $f(0)=0, f(1)=1$.

(1) 证明：存在 $0<c<1$，使得 $f(c)=\frac{1}{2}$；

(2) 证明：存在 $\xi\in(0,c), \eta\in(c,1)$，使得 $\frac{1}{f'(\xi)}+\frac{1}{f'(\eta)}=2$.

解：(1) 令 $\varphi(x)=f(x)-\frac{1}{2}$ $\therefore \varphi(0)=-\frac{1}{2}$, $\varphi(1)=\frac{1}{2}$

$\because \varphi(0)\varphi(1)<0$ $\therefore \exists c\in(0,1)$ $\varphi(c)=0$

即 $f(c)=\frac{1}{2}$

(2) $\exists \xi\in(0,c)$ $f'(\xi)=\frac{f(c)-f(0)}{c}=\frac{1}{2c}$

$\exists \eta\in(c,1)$ $f'(\eta)=\frac{f(1)-f(c)}{1-c}=\frac{1}{2(1-c)}$

$\therefore \frac{1}{f'(\xi)}+\frac{1}{f'(\eta)}=2$

例3.

$f(x)\in C[a,b]$. (a,b)内可导. $f(a)=f(b)$, $f'_+(a)>0$ 证

$\exists \xi, \eta\in(a,b)$. 使 $f'(\xi)>0$, $f'(\eta)<0$

解：1° $f'_+(a)>0 \Rightarrow \exists \delta>0$, $c=a+\delta$, $f(c)>f(a)$

2° $\exists \xi\in(a,c)$ $\eta\in(c,b)$

$f'(\xi)=\frac{f(c)-f(a)}{c-a}>0$ $f'(\eta)=\frac{f(b)-f(c)}{b-c}<0$

Case 2 含 ξ, η 的项复杂度不同

1° 留复杂中值项

2° $\begin{cases} (\cdots)' - L \\ \dfrac{(\ )'}{(\ )'} - C \end{cases}$

例 1.

例1 设 $f(x)\in C[a,b]$，在 (a,b) 内可导，且 $f(a)=f(b)=1$，证明：存在 $\xi,\eta\in(a,b)$，使得 $e^{\eta-\xi}[f'(\eta)+f(\eta)]=1$.

解: $e^{\eta-\xi}[f'(\eta)+f(\eta)]=1$

$\Rightarrow e^{\eta}[f'(\eta)+f(\eta)]=e^{\xi}$

$\hookrightarrow e^{x}f(x)-L$

证: 令 $\varphi(x)=e^{x}f(x)$ $\exists\,\eta\in(a,b)$，使 $\varphi'(\eta)=\dfrac{\varphi(b)-\varphi(a)}{b-a}$

$$e^{\eta}[f'(\eta)+f(\eta)]=\frac{e^{b}f(b)-e^{a}f(a)}{b-a}=\frac{e^{b}-e^{a}}{b-a}$$

令 $h(x)=e^{x}$. $\exists\,\xi\in(a,b)$ $\dfrac{h(b)-h(a)}{b-a}=h'(\xi)$

$\therefore \dfrac{e^{b}-e^{a}}{b-a}=e^{\xi}$

$\Rightarrow \dfrac{e^{\eta}[f'(\eta)+f(\eta)]}{e^{\xi}}=1$

例2 设 $f(x)\in C[a,b]$，在(a,b)内可导$(a>0)$，证明：存在 $\xi,\eta\in(a,b)$，使得

$$f'(\xi)=(a+b)\frac{f'(\eta)}{2\eta}.$$

解：

分析：$\dfrac{f'(\eta)}{2\eta}=\dfrac{f'(\xi)}{a+b}$

$\Downarrow$

$\dfrac{f'(\eta)}{(x^2)'}=C$

证 令 $g(x)=x^2$

$\therefore \exists\,\eta\in(a,b)$ 使 $\dfrac{f(a)-f(b)}{g(a)-g(b)}=\dfrac{f'(\eta)}{g'(\eta)}=\dfrac{f(a)-f(b)}{a^2-b^2}=\dfrac{f'(\eta)}{2\eta}$

$\exists\,\xi\in(a,b)$ 使 $\dfrac{f(a)-f(b)}{a-b}=f'(\xi)$

$\therefore f'(\xi)\dfrac{1}{a+b}=\dfrac{f'(\eta)}{2\eta}$

即 $(a+b)\dfrac{f'(\eta)}{2\eta}=f'(\xi)$

例3. $f(x)\in C[a,b]$，(a,b)内可导 $(a>0)$，证 $\exists\,\xi,\eta\in(a,b)$ 使 $abf'(\xi)=\eta^2f'(\eta)$

解：$x^2f'(x)=\dfrac{f'(x)}{(-\frac{1}{x})'}\to C$

令 $g(x)=-\frac{1}{x}$ $\therefore \exists\,\eta\in(a,b)$ $\dfrac{f(a)-f(b)}{\frac{1}{b}-\frac{1}{a}}=\dfrac{f'(\eta)}{\eta^2}=\dfrac{f(a)-f(b)}{a-b}\cdot ab$

$\exists\,\xi\in(a,b)$ $\dfrac{f(a)-f(b)}{a-b}=f'(\xi)$

即 $f'(\xi)ab=\eta^2f'(\eta)$

型六　L的使用.

① $f(b)-f(a)$ — L

② $f(a)$, $f(c)$, $f(a)$ 或 $f'(a)$, $f'(c)$, $f'(b)$

③ $f \Leftrightarrow f'$ (无积分)

例1. $\lim\limits_{x\to+\infty} x^2\left(\sin\frac{1}{x-1}-\sin\frac{1}{x+1}\right)$

解: 令 $f(t)=\sin t$　$f'(t)=\cos t$

$\exists \xi \in \left(\frac{1}{x+1}, \frac{1}{x-1}\right)$, $f'(\xi)=\dfrac{\sin\frac{1}{x-1}-\sin\frac{1}{x+1}}{\frac{1}{x-1}-\frac{1}{x+1}}$

$\Rightarrow f'(\xi)\,\frac{2}{x^2-1}=f\left(\frac{1}{x-1}\right)-f\left(\frac{1}{x+1}\right)$

$\Rightarrow$ 原式 $=\lim\limits_{\substack{x\to\infty\\ \xi\to 0}} \dfrac{2\cos\xi}{x^2-1}\cdot x^2$

$=\lim\limits_{\substack{x\to\infty\\ \xi\to 0}} \frac{2x^2}{x^2-1}=2$

例1 设 $\lim\limits_{x\to\infty} f'(x)=\mathrm{e}$, 且 $\lim\limits_{x\to\infty}[f(x)-f(x-1)]=\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^x$, 求 c.

解: $\lim\limits_{x\to\infty}\left[\frac{f(x)-f(x-1)}{x-(x-1)}\right]=\lim\limits_{x\to\infty} f'(\xi)=e$　$\xi\in(x-1,x)$

$\lim\limits_{x\to\infty}\left(\frac{x+c}{x-c}\right)^x=e^{\lim\limits_{x\to\infty}\frac{2cx}{x-c}}=e^{2c}=e$

$\therefore c=\frac{1}{2}$

例3. $f(0)=0$，$f''(x)>0$，证 $2f(1)<f(2)$

解：$\exists \xi_1 \in (0,1)$ $f'(\xi_1)=\dfrac{f(1)-f(0)}{1-0}$

$\exists \xi_2 \in (1,2)$ $f'(\xi_2)=\dfrac{f(2)-f(1)}{2-1}$

$\because f''(x)>0$ $\therefore f'(\xi_1)<f'(\xi_2)$

$\therefore f(1)-f(0)<f(2)-f(1)$ 而 $f(0)=0$

$\therefore 2f(1)<f(2)$

例4 设 $f(x)$ 在 $[a,b]$ 上可导，且 $|f'(x)|\leqslant M$，又 $f(x)$ 在 (a,b) 内至少有一个零点，证明：$|f(a)|+|f(b)|\leqslant M(b-a)$.

解：

1° $\exists c \in (a,b)$，使 $f(c)=0$

2° $f(c)-f(a)=f'(\xi_1)(c-a)$ $\xi_1 \in (a,c)$

$f(b)-f(c)=f'(\xi_2)(b-c)$ $\xi_2 \in (c,b)$

3° $\begin{cases} |f(a)|=|f'(\xi_1)|(c-a)\leq M(c-a) \\ |f(b)|\leq M(b-c) \end{cases}$

$\therefore |f(a)|+|f(b)|\leq M(b-a)$

Th 4. (Cauchy). 条件：$f(x)$ 在 $x=x_0$ 邻域内 $n+1$ 阶可导

结论

$$f(x) = P_n(x) + R_n(x)$$

其中 $P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$$R_n(x) = \begin{cases} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} & \text{— Lagrange 型} \\ o[(x-x_0)^n] & \text{— Peano 型} \end{cases}$$

If $x_0 = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

记：

① $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n(x)$

② $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$

③ $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots$

④ $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

⑤ $\frac{1}{1+x} = 1 - x + x^2 - x^3 \cdots$

⑥ $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$

⑦ $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$

例 1. $\lim_{x\to 0} \frac{e^{-x^2}-1+x^2}{x^3\sin x}$

解: $\lim_{x\to 0} \frac{e^{-x^2}-1+x^2}{x^4}$

$= \lim_{x\to 0} \frac{[1-x^2+\frac{1}{2!}(-x)^2-1+x^2]}{x^4} = 1/2$

例 2. $\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$

解: $(1+x)^\alpha = 1+\alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + o(x^2)$

$\therefore \sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2$

$\sqrt{1-x} = 1-\frac{1}{2}x-\frac{1}{8}x^2$

$\therefore \lim_{x\to 0} \frac{-\frac{1}{4}x^2}{x^2} = -1/4$

Part II 单调性与极值

一. defs

$y=f(x)$ $(x\in D)$, $x_0\in D$.

1. If $\exists \delta>0$, 当 $0<|x-x_0|<\delta$, 有 $f(x)<f(x_0)$
x_0 一极大点, $f(x_0)$ 一极大值

2. If $\exists \delta>0$, 当 $0<|x-x_0|<\delta$, 有 $f(x)>f(x_0)$
x_0 一极小点, $f(x_0)$ 一极小值

二. steps:

1° 找 x 范围, $x\in D$

2° 找 $f'(x)=\begin{cases}0\\ \text{不}\exists\end{cases}$

3°. 判别法:

Th1 (第一充分条件)

① $\begin{cases}x<x_0 & f'(x)>0\\ x>x_0 & f'(x)<0\end{cases}\Rightarrow x=x_0$ 极大

② $\begin{cases}x<x_0 & f'(x)<0\\ x>x_0 & f'(x)>0\end{cases}\Rightarrow x=x_0$ 极小

Th2 (第二充分条件)

设 $f'(x_0)=0$ 且 $f''(x)$ 连续.

① $f''(x_0)>0 \Rightarrow$ 极小

② $f''(x_0)<0 \Rightarrow$ 极大.

型一. 极值点判断.

1. $f'(1)=0$, $\lim\limits_{x\to 1}\dfrac{f'(x)}{\sin\pi x}=-1$, 判断 $x=1$ 为什么点?

解: 法一.

$\exists \delta>0$, 当 $0<|x-1|<\delta$ 时 $\dfrac{f'(x)}{\sin\pi x}<0$

$\begin{cases} f'(x)>0 & x\in(1,1+\delta) \\ f'(x)<0 & x\in(1-\delta,1) \end{cases} \Rightarrow x=1$ 为极小点,

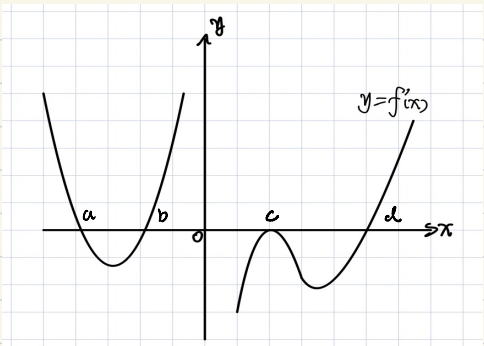
法二

$$-1=\lim_{x\to 1}\frac{f'(x)}{\sin\pi x}=\lim_{x\to 1}\frac{f'(x)}{\sin[\pi+\pi(x-1)]}=-\lim_{x\to 1}\frac{f'(x)}{\pi(x-1)}=-\frac{1}{\pi}\lim_{x\to 1}\frac{f'(x)}{x-1}$$

$\Rightarrow f'(1)=\pi>0 \Rightarrow x=1$ 为极小点,

2. $f(x) \in C(-\infty, +\infty)$

$y=f(x)$ 的图像如图所示，问 $f(x)$ 有几个极值点，



解：

1° $x \in (-\infty, +\infty)$

2° 怀疑对象 $x = a, b, 0, c, d$

3° $\begin{cases} x<a & f'(x)>0 \\ x>a & f'(x)<0 \end{cases}$ $x=a$ 极大

$\begin{cases} x<b & f'(x)<0 \\ x>b & f'(x)>0 \end{cases}$ $x=b$ 极小

$\begin{cases} x<0 & f'(x)>0 \\ x>0 & f'(x)<0 \end{cases}$ $x=0$ 极大

$\begin{cases} x<c & f'(x)<0 \\ x>0 & f'(x)<0 \end{cases}$

$\begin{cases} x<d & f'(x)<0 \\ x>d & f'(x)>0 \end{cases}$ $x=d$ 极小

例1 设 $f(x)$ 在 $x=0$ 处二阶可导，$f(0)=0$ 且 $\lim\limits_{x\to 0}\dfrac{f(x)+f'(x)}{x}=2$，则（ B ）.

(A) $f(0)$ 是 $f(x)$ 的极大值

(B) $f(0)$ 是 $f(x)$ 的极小值

(C) $(0,f(0))$ 是曲线 $y=f(x)$ 的拐点

(D) $f(0)$ 不是 $f(x)$ 的极值点，$(0,f(0))$ 也不是曲线 $y=f(x)$ 的拐点

解：

1° $\lim\limits_{x\to 0}\dfrac{f(x)+f'(x)}{x}=2$

$\Rightarrow \lim\limits_{x\to 0}f(x)+f'(x)=0 \Rightarrow f(0)+f'(0)=0$

$\Rightarrow f'(0)=0$

2° $\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x}+\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x}$

$=f'(0)+f''(0)=2 \Rightarrow f''(0)=2>0 \Rightarrow x=0$ 极小

例2 设 $f(x)$ 二阶连续可导，$\lim\limits_{x\to 2}\dfrac{f'(x)}{(x-2)^3}=\dfrac{2}{3}$，则（ A ）.

(A) $f(2)$ 是 $f(x)$ 的极小值

(B) $f(2)$ 是 $f(x)$ 的极大值

(C) $(2,f(2))$ 是曲线 $y=f(x)$ 的拐点

(D) $f(2)$ 不是函数 $f(x)$ 的极值点，$(2,f(2))$ 也不是曲线 $y=f(x)$ 的拐点

解：

$\lim\limits_{x\to 2}\dfrac{f'(x)}{(x-2)^3}=\dfrac{2}{3}>0 \Rightarrow f'(2)=0$

$\exists \delta>0$，当 $0<|x-2|<\delta$ 时，$\dfrac{f'(x)}{(x-2)^3}>0$

$\begin{cases} f'(x)<0 & x\in(2-\delta,2) \\ f'(x)>0 & x\in(2,2+\delta) \end{cases} \Rightarrow x=2$ 极小

型二. 函数零点/方程的根.

① $f(x) \in C[a,b]$, $f(a)f(b)<0 \Rightarrow \exists c \in (a,b)$, $f(c)=0$

② Rolle定理法.

$f(x) \Rightarrow F(x)$, $F'(x)=f(x)$

If $F(a)=F(b) \Rightarrow \exists c \in (a,b)$, $F'(c)=0 \Rightarrow f(c)=0$

例1 设 $a_0+\frac{a_1}{2}+\cdots+\frac{a_n}{n+1}=0$,证明:方程 $a_0+a_1x+\cdots+a_nx^n=0$ 至少有一个正根.

解.

令 $f(x)=a_0+a_1x+\cdots+a_nx^n$

$F(x)=a_0x+\frac{1}{2}a_1x^2+\cdots+\frac{a_n}{n+1}x^{n+1}$

$\because F(0)=F(1)=0 \quad \therefore \exists c \in (0,1) \quad F'(c)=0$

$\therefore \exists c \in (0,1) \quad f(c)=0$ 即至少有一正根

2. 证:方程 $e^x=x^2+4x+a$ 不可能有3个不同解

解: $f(x)=e^x+x^2-4x-a \quad x \in (-\infty,+\infty)$

反证:设 $\exists x_1<x_2<x_3$, $f(x_1)=f(x_2)=f(x_3)=0$

$\exists \xi_1 \in (x_1,x_2)$ $f'(\xi)=0 \Rightarrow e^{\xi_1}+2\xi_1-4=0$

$\exists \xi_2 \in (x_2,x_3)$ $f'(\xi_2)=0 \Rightarrow e^{\xi_2}+2\xi_2-4=0$

令 $h(x)=e^x+2x-4$ $h'(x)=e^x+2>0$ $\therefore h(x)$ 严格单调,有且仅有一个 x_0, $h(x_0)=0$

$\therefore \xi_1=\xi_2$ 与题设矛盾. 故不可能有3个解.

③ 单调性.

1° $f(x)=?$ $(?<x<?)$

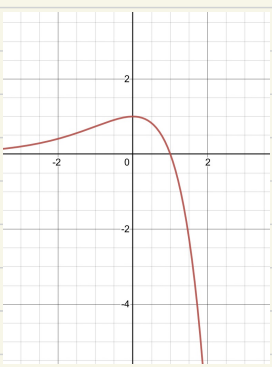
2° $f'(x)\begin{cases}=0\\ \text{无}\end{cases}\Rightarrow$ 极值点与极值.

3° 关注两侧走势. 作草图.

例2 讨论方程 $xe^{-x}=a(a>0)$ 根的个数.

解.

$f(x)=xe^{-x}$ $f'(x)=(1-x)e^{-x}$ $f'(x)=0$ 时. $x=1$



当 $x<1$ 时 $f'(x)>0$

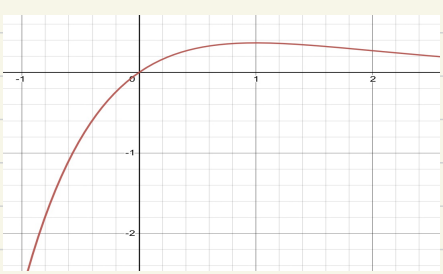
当 $x>1$ 时 $f'(x)<0$ $\Rightarrow f(x)$ 有极大值点

即 $f(x)$ 只有一个极大值. 则 $f(x)\le f(1)=\frac{1}{e}$

$\lim\limits_{x\to-\infty} xe^{x}=-\infty$

$\lim\limits_{x\to+\infty} xe^{-x}=0$

$\begin{cases}a>\frac{1}{e} & \text{无根}\\ a=\frac{1}{e} & \text{一根}\\ 0<a<\frac{1}{e} & \text{两根}\end{cases}$



3. 讨论 $\ln x = \frac{x}{e} - 2$ 有几根？

解：设 $f(x) = \ln x - \frac{x}{e} + 2 \quad (x>0)$

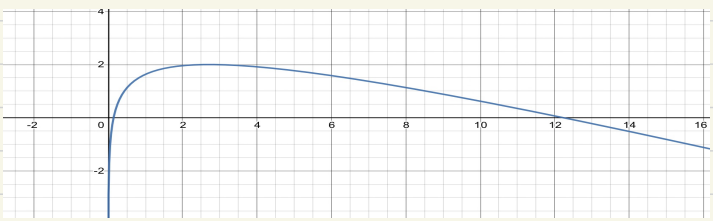
$f'(x) = \frac{1}{x} - \frac{1}{e}$ 若 $f'(x) = 0$，$x = e$

当 $x \in (0, e)$ 时，$f'(x) > 0$

$x \in (e, +\infty)$ 时，$f'(x) < 0$ $\Rightarrow x = e$ 为极大值点，

又 $f(e) = 2$ $\lim\limits_{x \to +\infty} f(x) = -\infty$ $\Rightarrow f(x) = 0$ 有两根

$\lim\limits_{x \to 0} f(x) = -\infty$



型三　不等式的证明

① 单调性

1. $x>0$，证 $\frac{x}{1+x}<\ln(1+x)<x$

解：令 $f(x)=x-\ln(1+x)$

$f(0)=0$，$f'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}$　$(x>0)$

$f'(x)>0$ 恒成立

$\therefore x-\ln(1+x)>0$ 恒成立

令 $g(x)=\ln(1+x)-\frac{x}{1+x}$

$g(0)=0$　$g'(x)=\frac{1}{1+x}-\frac{1+x-x}{(1+x)^2}$

$=\frac{x}{(1+x)^2}>0$

$\therefore g(x)>0$ 恒成立

即　$x>0$ 时，$\frac{x}{1+x}<\ln(1+x)<x$

法二：令 $f(t)=\ln(1+t)$　$f'(t)=\frac{1}{1+t}$

$\ln(1+x)=f(x)-f(0)=f'(\xi)x=\frac{x}{1+\xi}$　$(0<\xi<x)$

$\because \frac{x}{1+x}<\frac{x}{1+\xi}<x$

$\Downarrow$

$\frac{x}{1+x}<\ln(1+x)<x$

2. $e<a<b$. 证明 $a^b>b^a$

解:

证 $a^b>b^a \iff b\ln a - a\ln b > 0$

令 $f(x)=x\ln a - a\ln x \quad (x>a)$

$$\begin{cases} f(a)=0 \\ f'(x)=\ln a-\frac{a}{x}>0 \end{cases} \Rightarrow f(x)>0 \quad (x>a)$$

$\because b>a \quad \therefore f(b)>0 \Rightarrow$ 不等式成立.

3. $0<a<b$ 证: $\frac{\ln b-\ln a}{b-a} > \frac{2a}{a^2+b^2}$

证. 法1: $\frac{\ln b-\ln a}{b-a} > \frac{2a}{a^2+b^2} \iff (\ln b-\ln a)(a^2+b^2) > 2a(b-a)$ } 复杂!!

令 $f(x)=(\ln x-\ln a)(a^2+x^2)-2a(x-a) \quad f(a)=0 \quad (x>a)$

法2. 中值定理 { $\frac{f(b)-f(a)}{b-a}$ / $\frac{f(b)-f(a)}{g(b)-g(a)}$; f, f' 均在

令 $f(t)=\ln t \quad f'(t)=\frac{1}{t} \quad$ 左 $=\frac{f(b)-f(a)}{b-a}=f'(\xi)=\frac{1}{\xi} \quad (a<\xi<b)$

$\frac{1}{\xi}>\frac{1}{b}>\frac{2a}{a^2+b^2}$

4. 设 $b>a>0$，证明：$\ln\frac{b}{a}>\frac{2(b-a)}{a+b}$.

解：$(\ln b-\ln a)(a+b)-2(b-a)>0$

设 $f(x)=(\ln x-\ln a)(a+x)-2(x-a)$

$\therefore f'(x)=\frac{a+x}{x}+\ln x-\ln a-2=\frac{a}{x}+\ln x-1-\ln a\ (x>a)$

$f''(x)=-\frac{a}{x^2}+\frac{1}{x}=\frac{x-a}{x^2}>0\ (x>a)$

$\therefore f'(x)\uparrow$，$f'(a)=0$ $\therefore f'(x)>0$，$(x>a)$

$\therefore f(x)\uparrow$，$\therefore f(b)>f(a)=0$ 即原不等式成立.

Part III 几个小知识点

一. 凹凸性

(一) def. $y=f(x)$ $(x\in D)$

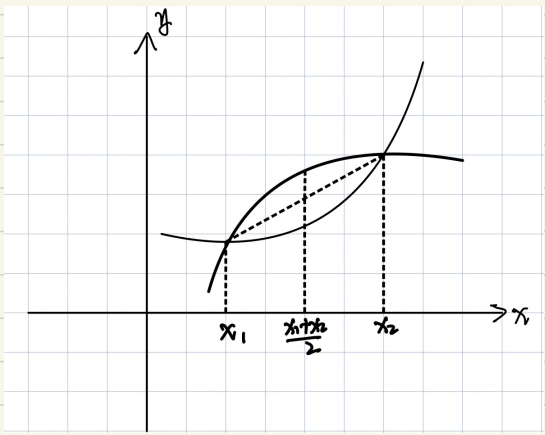
If $\forall x_1, x_2 \in D$ 且 $x_1 \neq x_2$

有 $f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$

称 $f(x)$ 在 D 上为凹函数.

若 $f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$

称 $f(x)$ 在 D 上为凸函数.



Note: $x=x_0$ 两侧凹凸性不同, $(x_0, f(x_0))$ 称为拐点.

(二) 判别法

Th. 设 $f(x) \in C[a,b]$, (a,b) 内二阶可导.

① If $f''(x) > 0$ $x \in (a,b)$ $\Rightarrow f(x)$ 在 $[a,b]$ 上凹

② If $f''(x) < 0$ $x \in (a,b)$ $\Rightarrow f(x)$ 在 $[a,b]$ 上凸

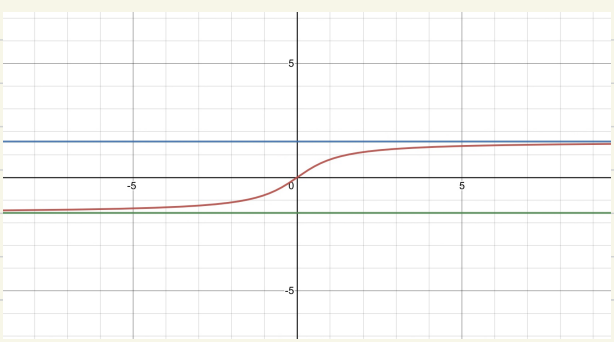
1. $f''(1)=0$，$f'''(1)=2$ 问 $(1,f(1))$ 是否为拐点.

解: $f''(1)=\lim\limits_{x\to 1}\dfrac{f''(x)}{x-1}=2>0$

$\Rightarrow \exists \delta>0$，$0<|x-1|<\delta$ 时，$\dfrac{f''(x)}{x-1}>0$

$$\begin{cases} f''<0 & x\in(1-\delta,1) \\ f''>0 & x\in(1,1+\delta) \end{cases} \Rightarrow (1,f(1)) \text{ 为 } y=f(x) \text{ 拐点}$$

二. 渐近线



$y=\arctan x$

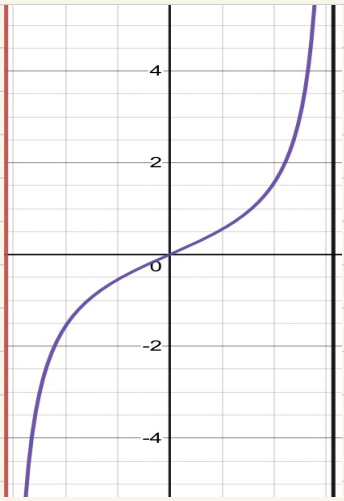
$y=\pi/2$

$y=-\pi/2$

1. 水平渐近线 If $\lim\limits_{x\to\infty}f(x)=A$，称 $y=A$ 为水平渐近线

2. 铅直渐近线：

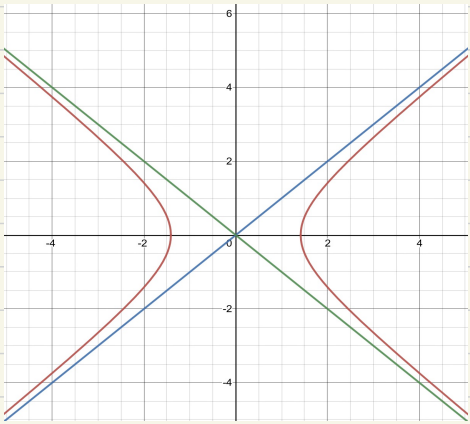
If $\begin{cases} \lim_{x\to a} f(x)=\infty \\ f(a-0)=\infty \\ f(a+0)=\infty \end{cases}$ 称 $x=a$ 为 铅直渐近线.



3. 斜渐近线

If $\lim_{x\to\infty} \frac{f(x)}{x} = A \quad (\neq 0, \infty)$

$\lim_{x\to\infty} [f(x)-Ax] = b$ $y=Ax+b$ 为斜渐近线.



水平与斜不能共存.

1. $y=\frac{2x^2+3x+2}{x-1}$ 求渐近线

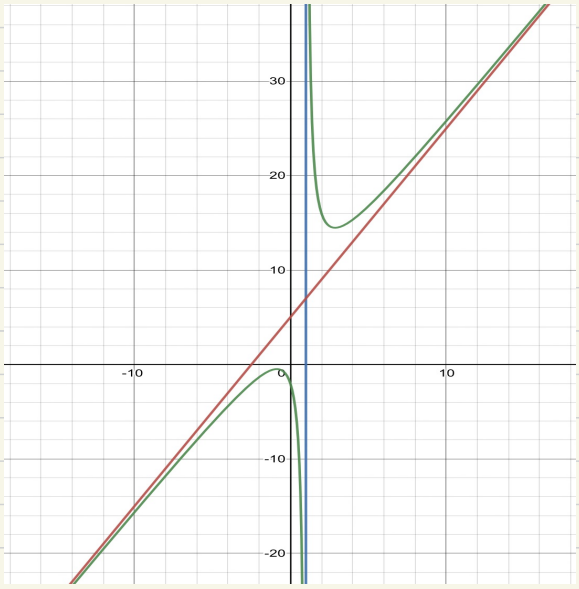
解：∵ $\lim_{x\to\infty} y=\infty$，无水平渐近线

$\lim_{x\to 1} y=\infty$ ∴ $x=1$ 为铅直

$\lim_{x\to\infty} \frac{y}{x}=2$

$\lim_{x\to\infty}(y-2x)=\lim_{x\to\infty}\frac{2x^2+3x+2-2x^2+2x}{x-1}=5$

∴ 斜渐近线 $y=2x+5$.



2. $y=\frac{x^2-x-2}{x^2-1}e^{\frac{1}{x}}$，求 $f(x)$ 渐近线

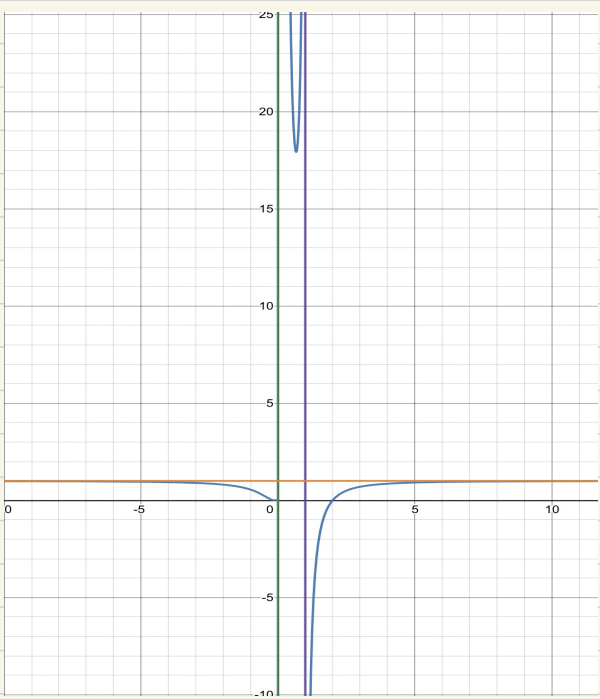
解：$\lim_{x\to 0^+} f(x)=\infty \Rightarrow x=0$ 为铅直.

$\lim_{x\to 0^-} f(x)=0$

$\lim_{x\to 1} f(x)=\infty \Rightarrow x=1$ 为铅直.

$\lim_{x\to\infty} f(x)=1 \Rightarrow y=1$ 为水平

$\lim_{x\to -1} f(x)=\frac{3}{2}e^{-1}$



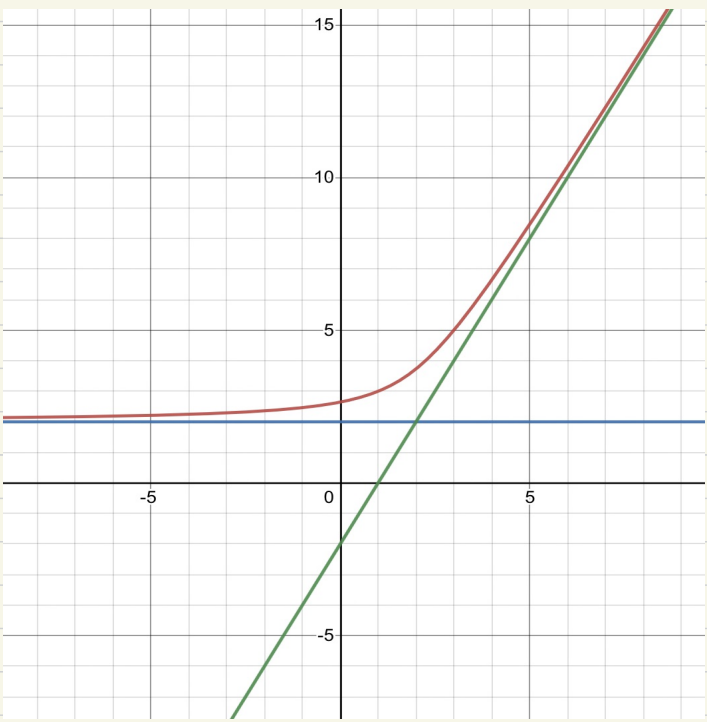
3. $y=f(x)=\sqrt{x^2-4x+7}+x$

解: $\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{-4x+7}{\sqrt{x^2-4x+7}-x}=2$ $y=2$ 为水平

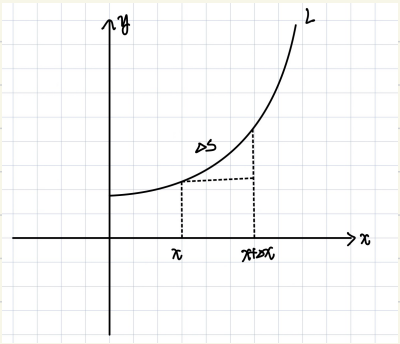
$\lim_{x\to+\infty} f(x)=+\infty$

$\lim_{x\to+\infty}\frac{f(x)}{x}=2$ $\lim_{x\to+\infty}[f(x)-2x]=-2$

$\therefore y=2x-2$ 为斜



三. 弧微分



$(\Delta s)^2 \doteq (\Delta x)^2 + (\Delta y)^2$

微分化 ↓

$(ds)^2 = (dx)^2 + (dy)^2$

1. $L: y = f(x) \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$

即 $ds = \sqrt{1 + f'^2(x)}\, dx$

2. $L = \begin{cases} x = \psi(t) \\ y = \phi(t) \end{cases} \Rightarrow ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = \sqrt{\psi'^2(t) + \phi'^2(t)}\, dt$

曲率

$$K = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}}$$

曲率半径 $\rho = \frac{1}{K}$.

第四章 不定积分

一. defs.

1. 原函数 — $f(x)$, $F(x)$. If $F'(x)=f(x)$, 则 $F(x)$ 为 $f(x)$ 原函数

Notes

① If $f(x)$ ∃ 原函数 $\Rightarrow$ ∃ 无数个原函数

② 任两个原函数相差常数

③ 连续函数 $\Rightarrow$ ∃ 原函数 ($\not\Leftarrow$)

2. 不定积分 — 设 $F(x)$ 为 $f(x)$ 的一个原函数, $F(x)+C$ 为所有原函数

称为不定积分. 记 $\int f(x)\,dx = F(x)+C$

二. 不定积分工具

(一) 基本公式.

(1) $\int k\,dx = kx + C$ (k 为常数).

(2) ① $\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C\ (a \neq -1)$; ② $\int \frac{dx}{x} = \ln|x| + C\ (x \neq 0)$.

(3) ① $\int a^x\,dx = \frac{a^x}{\ln a} + C\ (a>0, a\neq 1)$; ② $\int e^x\,dx = e^x + C$.

(4) ① $\int \sin x\,dx = -\cos x + C$; ② $\int \cos x\,dx = \sin x + C$;

③ $\int \tan x\,dx = -\ln|\cos x| + C$; ④ $\int \cot x\,dx = \ln|\sin x| + C$;

⑤ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$; ⑥ $\int \csc x\,dx = \ln|\csc x - \cot x| + C$;

⑦ $\int \sec^2 x\,dx = \tan x + C$; ⑧ $\int \csc^2 x\,dx = -\cot x + C$;

⑨ $\int \sec x \tan x\,dx = \sec x + C$; ⑩ $\int \csc x \cot x\,dx = -\csc x + C$.

(5) ① $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$; ② $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C\ (a>0)$;

③ $\int \frac{dx}{1+x^2} = \arctan x + C$; ④ $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C\ (a\neq 0)$;

⑤ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$; ⑥ $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + C$;

⑦ $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}| + C$;

⑧ $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$.

如. $\int \frac{1}{x^2-a^2}dx = \int \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx$

$= \frac{1}{2a}\int \frac{1}{x-a}d(x-a) - \frac{1}{2a}\int \frac{1}{x+a}d(x+a)$

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(二)积分法

1. 换元积分法

Case 1 第一类积分法

例 1 $\int xe^{x^2}dx$

解: 原 $= \frac{1}{2}\int e^{x^2}dx^2 = \frac{1}{2}e^{x^2}+C$

例 2. $\int \frac{x}{x^4+4}dx$

解: 原 $= \frac{1}{2}\int \frac{1}{x^4+4}dx^2$

$= \frac{1}{2}\cdot\frac{1}{2}\operatorname{arctan}\left(\frac{x^2}{2}\right)+C$

$= \frac{1}{4}\operatorname{arctan}\left(\frac{x^2}{2}\right)+C$

3. $\int \frac{dx}{\sqrt{x}(1+x)}$

解: 原 $= 2\int \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}dx$

$= 2\int \frac{1}{1+(\sqrt{x})^2}d\sqrt{x}$

$= 2\arctan(\sqrt{x}) + C$

即 $\int f[\varphi(x)]\varphi'(x)dx = \int f[\varphi(x)]d\varphi(x)$

令 $\varphi(x) = t \Rightarrow \int f(t)dt = F(t)+C = F[\varphi(x)]+C$

4. $\int \frac{x}{(2x+3)^2}dx$

解: $\frac{1}{4}\int \frac{(2x+3)-3}{(2x+3)^2}d(2x+3)$

$2x+3=t$ $\frac{1}{4}\int \frac{t-3}{t^2}dt = \frac{1}{4}\int \frac{1}{t}-\frac{3}{t^2}dt$

$= \frac{1}{4}\ln t + \frac{3}{4t} + C$

即原 $= \frac{1}{4}\ln(2x+3) + \frac{3}{8x+12} + C$

5. $\int \frac{x-1}{x^2+2x+2}dx$

解: $\frac{1}{2}\int \frac{(2x+2)-4}{x^2+2x+2}dx$

$= \frac{1}{2}\int \frac{d(x^2+2x+2)}{x^2+2x+2} - 2\int \frac{d(x+1)}{1+(x+1)^2}$

$= \frac{1}{2}\ln(x^2+2x+2) - 2\arctan(x+1) + C$

Case 2. 第二类换元积分法

① 无理 $\Rightarrow$ 有理

1. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$

解: $2\int \frac{d\sqrt{x}}{1+(\sqrt{x})^2} = 2\arctan\sqrt{x}+C$

2. $\int \sqrt{\frac{1-x}{1+x}}\,dx$

解: $\int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= \arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}}\,dx = \arcsin x + \int \frac{d(1-x^2)}{2\sqrt{1-x^2}}$

$= \arcsin x + \sqrt{1-x^2} + C$

3. $\int \frac{dx}{1+\sqrt{x}}$

解: 令 $x=t^2$ $\int \frac{dt^2}{1+t} = \int \frac{2t\,dt}{1+t} = 2\int (1-\frac{1}{1+t})\,dt$

$= 2[t-\ln(1+t)]+C$

$\Rightarrow$ 原 $= 2[\sqrt{x}-\ln(1+\sqrt{x})]+C$

原理: $\int f(x)\,dx \xrightarrow{x=\varphi(t)} \int f[\varphi(t)]\varphi'(t)\,dt = \int g(t)\,dt = G(t)+C$

$= G[\varphi^{-1}(x)]+C$.

② 三角代换

$\sqrt{a^2-x^2}$ $x=a\sin t$ $a\cos t$

$\sqrt{x^2+a^2}$ $x=a\tan t$ $a\sec t$.

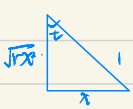
$\sqrt{x^2-a^2}$ $x=a\sec t$ $a\tan t$.

1. $\int \frac{dx}{x^2\sqrt{1+x^2}}$

解: 设 $x=\sin t$ $\int \frac{\cos t\,dt}{\sin^2 t\cos t} = \int \frac{1}{\sin^2 t}dt = \int \csc^2 t\,dt$

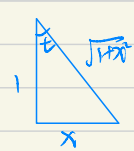
$= -\cot t + C$

$= \frac{\sqrt{1+x^2}}{x} + C$



2. $\int \frac{1}{\sqrt{(x^2+1)^3}}dx$

解: $x=\tan t$ $= \int \frac{1}{\sec^3 t}d\tan t$



$= \int \frac{\sec^2 t}{\sec^3 t}dt = \int \frac{1}{\sec t}dt = \int \cos t\,dt$

$= \sin t + C$

$= \frac{x}{\sqrt{1+x^2}} + C$

2. 分部积分法

$$(uv)' = u'v + uv'$$

$$\Rightarrow \int (uv)'dx = \int u'v\,dx + \int uv'dx$$

$$\Rightarrow uv = \int v\,du + \int u\,dv.$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

① $\int$ 幂 $\times$ 指 dx (指幂)

1. $\int x^2 e^x dx = \int x^2\,de^x$

$= x^2e^x - \int 2x\,e^x\,dx$

$= x^2e^x - \int 2x\,de^x = x^2e^x - 2\left(xe^x - \int e^x dx\right)$

$= x^2e^x - 2xe^x + 2e^x + C$

$= (x^2 - 2x + 2)e^x + C$

② $\int$ 幂 $\times$ 对数 dx (幂对)

2. $\int x \ln x\,dx = \int \ln x\,d\frac{x^2}{2}$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot 2\ln x \cdot \frac{1}{x}\,dx$

$= \frac{x^2}{2}\ln x - \int x \ln x\,dx$

$= \frac{x^2}{2}\ln x - \int \ln x\,d\frac{x^2}{2}$

$= \frac{x^2}{2}\ln x - \frac{x^2}{2}\ln x + \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}(\ln x - \ln x) + \frac{1}{4}x^2 + C$

→必须是1次

$\int$ 幂 $\times$ 三角 dx （番幂）

3. $\int x^2\cos 2x\,dx = \int x^2\, d(\frac{1}{2}\sin 2x)$

$= \frac{x^2}{2}\sin 2x - \int 2x\cdot\frac{1}{2}\sin 2x\,dx$

$= \frac{x^2}{2}\sin 2x - \int x\sin 2x\,dx = \frac{x^2}{2}\sin 2x + [\int x\, d(\frac{1}{2}\cos 2x)]$

$= \frac{x^2}{2}\sin 2x + [\frac{x}{2}\cos 2x - \int \frac{1}{2}\cos 2x\,dx]$

$= \frac{x^2}{2}\sin 2x + \frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$

4. $\int x\sin^2 x\,dx = \frac{1}{2}\int x(1-\cos 2x)\,dx$

解：

$= \frac{x^2}{4} - \frac{1}{2}\int x\cos 2x\,dx$

$= \frac{x^2}{4} - \frac{1}{2}\int x\,d(\frac{1}{2}\sin 2x)$

$= \frac{x^2}{4} - \frac{1}{2}[\frac{x}{2}\sin x - \int \frac{1}{2}\sin 2x\,dx]$

$= \frac{x^2}{4} - \frac{x}{4}\sin x - \frac{1}{8}\cos 2x + C$

5. $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx$

解：$\int x\,d(\tan x) - \frac{1}{2}x^2 + C$

$= x\tan x - \int \tan x\,dx - \frac{1}{2}x^2 + C$

$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$

㈣ $\int$ 幂 × 反三角 dx (指 arc·)

6. $\int x^2 \operatorname{arctan} x\,dx = \int \operatorname{arctan} x \cdot d(\frac{1}{3}x^3)$

$= \frac{x^3}{3}\operatorname{arctan} x - \frac{1}{3}\int x^3 \cdot \frac{1}{1+x^2}\,dx$

$= \frac{x^3}{3}\operatorname{arctan} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

$= \frac{x^3}{3}\operatorname{arctan} x - \frac{1}{3}\int \frac{x(1+x^2)-x}{1+x^2}\,dx$

$= \frac{x^3}{3}\operatorname{arctan} x - \frac{1}{3}\int x - \frac{x}{1+x^2}\,dx$

$= \frac{x^3}{3}\operatorname{arctan} x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C$

7. $\int \arcsin x\,dx$

解: 原 $= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\arcsin x + \int \frac{-x}{\sqrt{1-x^2}}\,dx \quad = x\arcsin x + \int \frac{d(1-x^2)}{2\sqrt{1-x^2}}$

$= x\arcsin x + \sqrt{1-x^2} + C$

⑤ $\int e^{ax} \times \begin{cases} \cos bx \\ \sin bx \end{cases} dx$

8. $\int e^x \cos 2x \, dx$

解: 令 $I = \int e^x \cos 2x \, dx = \int \cos 2x \, de^x$

$= e^x \cos 2x - \int e^x (-2\sin 2x) dx$

$= e^x \cos 2x + 2\int \sin 2x \, de^x$

$= e^x \cos 2x + 2\left[e^x \sin 2x - \int e^x 2\cos 2x \, dx\right]$

$= e^x \cos 2x + 2e^x \sin 2x - 4\int e^x \cos 2x \, dx$

$\therefore I = e^x \cos 2x + 2e^x \sin 2x - 4I$

$\therefore I = \frac{1}{5}(e^x \cos 2x + 2e^x \sin 2x) + C$

⑥ $\begin{cases} \int \sec^n x\,dx \\ \int \csc^n x\,dx \end{cases}$ 其中 n 为奇数.

9. 求 $\int \sec^3 x\,dx$

解: 令 $I=\int \sec^3 x\,dx$

$$=\int \sec x\, d(\tan x)$$

$$=\sec x\tan x-\int \tan^2 x\sec x\,dx$$

$$=\sec x\tan x-\int (\sec^2 x-1)\sec x\,dx$$

$$=\sec x\tan x-I+\ln|\sec x+\tan x|$$

$$\Rightarrow I=\frac{\sec x\tan x+\ln|\sec x+\tan x|}{2}+C$$

二. 特殊函数的不定积分

(一) 有理函数的不定积分

def - $R(x)=\frac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ — 多项式

① If $\deg P < \deg Q$, $R(x)$ — 真分式

② If $\deg P \geq \deg Q$, $R(x)$ — 假分式

原则:

原则一: If $R(x)$ 一假分式. $R(x)=$ 多项式 + 真分式

1. $R(x)=\frac{3x^4+x^2-3}{x^2+x+1}=(3x^2-3x+1)+\frac{2x-4}{x^2+x+1}$

原则二. If $R(x)$ - 真分式 $\begin{cases}\text{分子不动}\\ \text{分母因式分解}\end{cases}\Rightarrow$ 拆成部分和.

例 do

① $\frac{x^3-2x^2+1}{(2x-1)(x+1)^3}=\frac{A}{2x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}$

$A(x+1)^3+B(2x-1)(x+1)^2+C(2x-1)(x+1)+D(2x-1)=x^3-4x^2+1$

② $\frac{x^2-3x+1}{(2x+1)(x^2+1)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+1}$

④ $\frac{3x^2-1}{x^2(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$

1.

(1) $\int \frac{1}{x^2+x-12}\,dx = \frac{1}{7}\left[\int \frac{1}{x-4}\,dx - \int \frac{1}{x+3}\,dx\right]$

$= \frac{1}{7}\ln(x-4) - \frac{1}{7}\ln(x+3)$

$= \frac{1}{7}\ln\frac{x-4}{x+3} + C$

(2) $\int \frac{1}{x^2+2x+2}\,dx$

$= \int \frac{1}{(x+1)^2+1}\,d(x+1)$

$= \arctan(x+1) + C$

(3) $\int \frac{5x-1}{x^2-x-2}\,dx$

解: $x^2-x-2 = (x-2)(x+1)$

$\begin{cases} A+B=5 \\ -2A+B=-1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \end{cases}$

$\therefore \int \frac{5x-1}{x^2-x-2}\,dx = \int \left(\frac{3}{x-2} + \frac{2}{x+1}\right)dx$

$= 3\ln(x-2) + 2\ln(x+1) + C$

(4) $\int \frac{x+2}{x^2+x+1}\,dx$

$= \int \frac{\frac{1}{2}(2x+1)+3}{x^2+x+1}\,dx$

$= \frac{1}{2}\int \frac{d(x^2+x+1)}{x^2+x+1} + \frac{3}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$= \frac{1}{2}\ln(x^2+x+1) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x+\frac{1}{2})$

$= \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}(x+\frac{1}{2}) + C$

1. $\int \frac{3x+2}{x(1+x^2)}\,dx$

解: $\frac{3x+2}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\Rightarrow \begin{cases} A+B=0 \\ A=2 \\ C=3 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=2 \\ C=3 \end{cases}$

$\therefore \int \frac{3x+2}{x(1+x^2)} = \int \frac{-2}{x}\,dx + \int \frac{2x+3}{x^2+1}\,dx$

$= -2\ln|x| + \int \frac{d(x^2)}{x^2+1} + \int \frac{3}{x^2+1}\,dx$

$= -2\ln|x| + \ln(x^2+1) + 3\arctan x + C$

2. $\int \frac{x^3+3}{x^2(1+x)}\,dx$

解: $\frac{x^3+3}{x^2(1+x)} = \frac{x^3+3-x^2-x}{x^2(1+x)} + 1 = 1 + \frac{3-x^2}{x^2(1+x)}$

$\frac{3-x}{x^2(1+x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1+x}$

$= \frac{A(1+x)+Bx(1+x)+Cx^2}{x^2(1+x)}$

$\Rightarrow \begin{cases} A+C=-1 \\ A+B=0 \\ B=3 \end{cases} \Rightarrow \begin{cases} A=-3 \\ B=3 \\ C=2 \end{cases}$

$\Rightarrow \int \frac{x^3+3}{x^2(1+x)} = \int 1 + \frac{3-x^2}{x^2(1+x)}\,dx = x + \int \frac{-3}{x}\,dx + \int \frac{3}{x^2}\,dx + \int \frac{2}{1+x}\,dx$

$= x - 3\ln|x| - \frac{3}{x} + 2\ln|1+x| + C.$

3. ① $\int \frac{dx}{x(x^4+3)}$ ② $\int \frac{x^2+1}{x^4+1}dx$

① 解 $\int \frac{dx}{x(x^4+3)} = \int \frac{x^3dx}{x^4(x^4+3)} = \frac{1}{4}\int \frac{dx^4}{x^4(x^4+3)}$

$\overset{x^4=t}{=\!=} \frac{1}{4}\int \frac{dt}{t(t+3)} = \frac{1}{12}\int \frac{1}{t} - \frac{1}{t+3} = \frac{1}{12}\ln\left|\frac{t}{t+3}\right| + C$

$\Rightarrow \frac{1}{12}\ln\left|\frac{x^4}{x^4+3}\right| + C$

②解: $\int \frac{x^2+1}{x^4+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+(\sqrt{2})^2}$

$\overset{x-\frac{1}{x}=t}{=\!=} \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\operatorname{arctan}\frac{t}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}\operatorname{arctan}\frac{x-\frac{1}{x}}{\sqrt{2}} + C.$

(二) 无理函数的不定积分

①无理化为有理函数

1. $\int \frac{dx}{\sqrt{x}(1+x)}$

解: $2\int \frac{dx}{2\sqrt{x}} \cdot \frac{1}{1+x}$

$= 2\int \frac{1}{1+(\sqrt{x})^2} d(\sqrt{x}) = 2\arctan\sqrt{x} + C$

2. $\int \frac{dx}{\sqrt{x-x^2}}$

解: $\int \frac{1}{\sqrt{x}\sqrt{1-x}} dx$

$= 2\int \frac{dx}{2\sqrt{x}} \cdot \frac{1}{\sqrt{1-x}} = 2\int \frac{1}{\sqrt{1-x}} d(\sqrt{x}) = 2\int \frac{1}{\sqrt{1-(\sqrt{x})^2}} d(\sqrt{x})$

$= 2\arcsin\sqrt{x} + C$

3. $\int \sqrt{\frac{1+x}{1-x}} dx$

解: $\int \frac{1+x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx$

$= \arcsin x - \sqrt{1-x^2} + C$

① 需要化为有理数.

4. $\int e^{\sqrt{x}}\,dx$

解: 令 $\sqrt{x}=t$ $\quad dx=2t\,dt$

$$\int e^{t}\cdot 2t\,dt = \int 2t\,de^{t}$$

$$= 2te^{t} - \int 2e^{t}\,dt$$

$$= 2te^{t} - 2e^{t} = (2t-2)e^{t} \Rightarrow (2\sqrt{x}-2)e^{\sqrt{x}}+C$$

5. $\int \frac{dx}{1+\sqrt{x}}$

解: 令 $\sqrt{x}=t$ $\quad$ 原式 $=\int \frac{2t\,dt}{1+t} = 2\int 1-\frac{1}{1+t}\,dt$

$$= 2t-2\ln(1+t) \Rightarrow 2\sqrt{x}-2\ln(1+\sqrt{x})+C$$

(三) 三角有理函数不定积分

1. $\int \frac{dx}{1+\cos x}$

解: 法一. $\int \frac{dx}{1+\cos x} = \int \frac{1}{2\cos^2\frac{x}{2}}\,dx = \int \sec^2\frac{x}{2}\,d(\frac{x}{2})$

$= \tan\frac{x}{2} + C$

法二. $\int \frac{1-\cos x}{\sin^2 x}\,dx = \int \csc^2 x\,dx - \int \frac{d(\sin x)}{\sin^2 x}$

$= -\cot x + \frac{1}{\sin x} + C$

2. $\int \frac{dx}{1+\sin x}$

解: 法一 $\int \frac{d(x-\frac{\pi}{2})}{1+\cos(x-\frac{\pi}{2})} = \tan(\frac{x}{2}-\frac{\pi}{4}) + C$

法二 $\int \frac{1-\sin x}{\cos^2 x}\,dx = \int \sec^2 x\,dx + \int \frac{d\cos x}{\cos^2 x}$

$= \tan x - \frac{1}{\cos x} + C$

3. $\int \frac{dx}{\sin x+\cos x}$

解: $\frac{1}{\sqrt{2}}\int \frac{d(x-\frac{\pi}{4})}{\cos(x-\frac{\pi}{4})} = \frac{1}{\sqrt{2}}\int \sec(x-\frac{\pi}{4})\, d(x-\frac{\pi}{4})$

$= \frac{1}{\sqrt{2}} \ln\left|\sec(x-\frac{\pi}{4}) + \tan(x-\frac{\pi}{4})\right| + C$

4. $\int \tan^3 x \sec x\, dx$

解: 原 $= \int \tan^2 x\, d\sec x$

$= \int \sec^2 x - 1\, d\sec x$

$= \frac{1}{3}\sec^3 x - \sec x + C$

第五章 定积分及其应用

第一节：概念与性质

一. def.

设 $f(x)$ 在 $[a,b]$ 上有界

① $a = x_0 < x_1 < \cdots < x_n = b$

$[a,b] = [x_0, x_1] \cup \cdots [x_{n-1}, x_n]$

② $\forall \xi_i \in [x_{i-1}, x_i]$，作 $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$

③ 令 $\lambda = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$ If $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ $\exists$，称 $f(x)$ 在 $[a,b]$ 上可积，且极限值为 $f(x)$ 在 $[a,b]$ 上定积分

记 $\int_a^b f(x)dx$.

即 $\int_a^b f(x)dx = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$

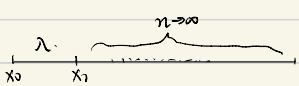
Notes

① $\begin{cases} f(x) \in C[a,b] \Rightarrow f(x) \text{ 在 } [a,b] \text{ 上可积} \\ f(x) \text{ 在 } [a,b] \text{ 上除有限第一类间断点连续，可积} \end{cases}$

② $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ 与 $\begin{cases} [a,b] \text{ 的分法} \\ \xi_i \text{ 的取法} \end{cases}$ 均无关

③ $f(x)$ 在 $[a,b]$ 上有界 不一定可积，（如狄利克雷函数）

④ $\lambda \to 0 \Rightarrow n \to \infty$，$\not\Leftarrow$



⑤ 设 $f(x)$ 在 $[0,1]$ 上可积，$\Rightarrow \int_0^1 f(x)dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\frac{i}{n})$

例1 计算 $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\cdots+n^2}{n^3}$.

解. 原式 $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^2=\int_0^1 x^2\,dx=\frac{1}{3}$

例2 计算 $\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{n+n}\right)$.

解. 原式 $=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{1+\frac{i}{n}}=\int_0^1\frac{1}{1+x}\,dx$

$=\ln(1+x)\Big|_0^1=\ln 2$

三. 性质 (一般性质)

① $\int_a^b dx=b-a$

② i) If $f(x)\geq 0$ $x\in[a,b]$ $\Rightarrow\int_a^b f(x)\,dx$

ii) If $f(x)\geq g(x)$ $x\in[a,b]$ $\Rightarrow\int_a^b f(x)\,dx\geq\int_a^b g(x)\,dx$

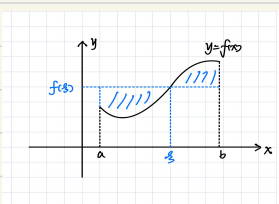
iii) If $f(x)$, $|f(x)|$ $x\in[a,b]$ 上可积,

则 $\left|\int_a^b f(x)\,dx\right|\leq\int_a^b|f(x)|\,dx$ ☆

③ 积分中值定理: 设 $f(x)\in C[a,b]$

则 $\exists\,\xi\in[a,b]$, 使

$f(\xi)(b-a)=\int_a^b f(x)\,dx$.



证: $\because f(x)\in C[a,b]$

$\therefore f(x)$ 在 $[a,b]$ 上 $\exists\ m,M$. $\therefore m\leq f(x)\leq M$

$\Rightarrow\int_a^b m\,dx\leq\int_a^b f(x)\,dx\leq\int_a^b M\,dx$

$\Rightarrow m(b-a)\leq\int_a^b f(x)\,dx\leq M(b-a)\Rightarrow m\leq\frac{\int_a^b f(x)\,dx}{b-a}\leq M$

$\therefore\exists\,\xi\in[a,b]$, 使 $f(\xi)=\int_a^b f(x)\,dx\Big/(b-a)$

$\Rightarrow(b-a)f(\xi)=\int_a^b f(x)\,dx$

积分中值定理推广

设 $f(x) \in C[a,b]$，则 $\exists \xi \in (a,b)$，使 $f(\xi)(b-a) = \int_a^b f(x)dx$.

证：令 $F(x) = \int_a^x f(t)dt$，$F'(x) = f(x)$

$F(a) = 0$，$F(b) = \int_a^b f(t)dt = \int_a^b f(x)dx$

$\therefore \int_a^b f(x) = F(b) - F(a)$ $\xrightarrow{L}$ $\exists \xi \in (a,b)$，$F'(\xi) = \frac{F(b)-F(a)}{b-a}$

即 $\exists \xi \in (a,b)$，$(b-a)F'(\xi) = F(b) - F(a)$

而 $F'(\xi) = f(\xi)$，$F(b) - F(a) = \int_a^b f(x)dx$ $\Rightarrow f(\xi)(b-a) = \int_a^b f(x)dx$.

1. $f(x) \in C[0,1]$，$(0,1)$ 内可导，$f(0) = \int_0^1 f(x)dx$. 证 $\exists \xi \in (0,1)$，使 $f'(\xi) = 0$

解：令 $F(x) = \int_0^x f(t)dt$

$\therefore F'(x) = f(x)$，$F(1) = \int_0^1 f(x)dx$ $F(0) = 0$

$\therefore \int_0^1 f(x)dx = F(1) - F(0) = F'(c)(1-0) = f(c)$ $c \in (0,1)$

$\because f(0) = f(c)$ $\therefore \exists \xi \in (0,c) \subset (0,1)$，使 $f'(\xi) = 0$.

第二节 基本理论

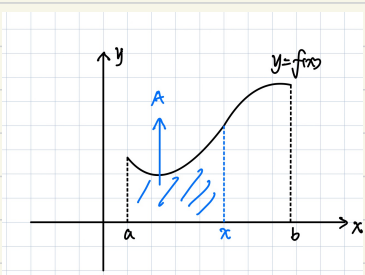
说明：

定积分的结果只与上、下限以及函数关系有关，与变量名无关

L：$y = f(x) \geq 0$，$x \in (a,b)$，$\forall x \in [a,b]$

$A = \int_a^x f(t)dt = \Phi(x)$

积分上限函数



Notes

① $\int_a^x f(x)\,dx$，表达式中的 x 与上限中 x 不同.

↳ $\int_a^x f(t)\,dt$.

② $\int_a^x f(x+t)\,dt$. 此时表达式中的 x 与上限中 x 相同

Th1. 设 $f(x) \in C[a,b]$，$\Phi(x) = \int_a^x f(t)\,dt$，则 $\Phi'(x) = f(x)$

证. $\Delta\Phi = \Phi(x+\Delta x) - \Phi(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt$

$= \int_x^{x+\Delta x} f(t)\,dt$ $\because f(x) \in C[a,b]$

$\therefore \Delta\Phi = f(\xi)\Delta x$. $(x < \xi < x+\Delta x)$ $\Rightarrow \frac{\Delta\Phi}{\Delta x} = f(\xi)$ $\xi \in (x, x+\Delta x)$

$\lim_{\Delta x \to 0} \frac{\Delta\Phi}{\Delta x} = f(x)$ 即 $\Phi'(x) = f(x)$

③ 若 $f(x) \in C[a,b] \Rightarrow \int_a^x f(t)\,dt$ 可导.

④ 若 $f(x) \in C[a,b] \Rightarrow f(x)$ 有原函数

⑤ $\int_a^{\varphi(x)} f(t)\,dt$，其中 $\varphi(x)$ 可导. $\Rightarrow \frac{d\int_a^{\varphi(x)} f(t)\,dt}{dx} = f[\varphi(x)]\varphi'(x)$

⑥ $\int_{\psi_1(x)}^{\psi_2(x)} f(t)\,dt$. $\psi_1(x)$，$\psi_2(x)$ 均可导 $\Rightarrow \frac{d\int_{\psi_1(x)}^{\psi_2(x)} f(t)\,dt}{dx} = f[\psi_2(x)]\psi_2'(x) - f[\psi_1(x)]\psi_1'(x)$

例 1.

例1 求极限 $\lim_{x\to 0} \dfrac{\int_0^x e^{-t^2}\cos t\,dt - x}{x^3}$

解. 原式 $\lim_{x\to 0} \frac{e^{-x^2}\cos x - 1}{3x^2}$

$= \lim_{x\to 0} \frac{e^{-x^2}(\cos x - 1) + e^{-x^2} - 1}{3x^2}$

$= \lim_{x\to 0} \frac{\cos x - 1 + e^{-x^2} - 1}{3x^2} = \lim_{x\to 0} \frac{-\frac{1}{2}x^2}{3x^2} = -\frac{1}{2}$.

例2. $f(x)$ 连续. $f(0)=0$, $f'(0)=6$, 求 $\lim\limits_{x\to 0}\dfrac{\int_0^x (x-t)f(t)\,dt}{x^2}$

解. 原式 $=\lim\limits_{x\to 0}\dfrac{x\int_0^x f(t)\,dt-\int_0^x tf(t)\,dt}{x^3}$

$=\lim\limits_{x\to 0}\dfrac{xf(x)+\int_0^x f(t)\,dt-xf(x)}{3x^2}=\lim\limits_{x\to 0}\dfrac{\int_0^x f(t)\,dt}{3x^2}$

$=\lim\limits_{x\to 0}\dfrac{f(x)}{6x}\Rightarrow \lim\limits_{x\to 0}\dfrac{f'(x)}{6}=1$

即 原式为 1.

Th2. (N-L) 设 $f(x)\in C[a,b]$, $F(x)$ 为 $f(x)$ 一个原函数

则有 $\int_a^b f(x)dx=F(b)-F(a)$

证. 令 $\Phi(x)=\int_a^x f(t)dt$ $\because \Phi'(x)=f(x)$, $F'(x)=f(x)$

$\therefore F(x)-\Phi(x)\equiv C_0\Rightarrow F(a)-\Phi(a)\equiv C_0$, $F(b)-\Phi(b)\equiv C_0$

$\Rightarrow F(a)-\Phi(a)=F(b)-\Phi(b)$

$\because \Phi(a)=0$, $\Phi(b)=F(b)-F(a)\Rightarrow$ 即 $\int_a^b f(t)dt=F(b)-F(a)$

二. 定积分的特殊性质

1. 对称性质

$f(x) \in C[-a,a]$，则 $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$

证. 左 $\int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$.

$\int_{-a}^{0} f(x)dx \xrightarrow{x=-t} \int_a^0 f(-t)(-dt) = \int_0^a f(-t)dt$

$\therefore$ 左 $= \int_0^a [f(t)+f(-t)]dt$.

例1. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin x+1}$

解: $I = \int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\sin x} + \frac{1}{1-\sin x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x}dx = 2\int_0^{\frac{\pi}{4}} d\tan x = 2\tan x\Big|_0^{\frac{\pi}{4}} = 2$

2. 三角函数性

If $f(x) \in C[0,1]$，则 $\int_0^{\frac{\pi}{2}} f(\sin x) = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

(PS: 一般遇到 $\int_a^b \cdots = \int_0^b \cdots$，教练为令 $x+t=a+b$)

证. $\int_0^{\frac{\pi}{2}} f(\sin x)dx \xrightarrow{x+t=\frac{\pi}{2}} \int_{\frac{\pi}{2}}^0 f(\cos t)(-dt)$

$\Rightarrow \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos t)dt \Rightarrow \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$

例2. $I = \int_0^1 \frac{dx}{x+\sqrt{1-x^2}}$

解: 令 $x = \sin t$ $I = \int_0^{\frac{\pi}{2}} \frac{\cos t\,dt}{\sin t+\cos t} = \int_0^{\frac{\pi}{2}} \frac{\cos x\,dx}{\sin x+\cos x}$

$\because \int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx \quad \therefore \int_0^{\frac{\pi}{2}} \frac{\sin x\,dx}{\cos x+\sin x} = I$

$\therefore 2I = \int_0^{\frac{\pi}{2}} dx \Rightarrow I = \frac{\pi}{4}$

特别地

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \triangleq I_n \quad \begin{cases} I_n = \frac{n-1}{n} I_{n-2} \\ I_0 = \frac{\pi}{2} \\ I_1 = 1 \end{cases}$$

故 $\int_0^{\frac{\pi}{2}} \sin^8 x\,dx = I_8 = \frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}} \cos^{11} x\,dx = I_{11} = \frac{10}{11}\times\frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1$

例3. $I = \int_0^2 x^2\sqrt{2x-x^2}\,dx$

解: $I = \int_0^2 x^2\sqrt{1-(x-1)^2}\,d(x-1)$

$= \int_0^2 [1+(x-1)]^2\sqrt{1-(x-1)^2}\,d(x-1)$

$= \int_{-1}^1 (1+x)^2\sqrt{1-x^2}\,dx = \int_{-1}^1 (x^2+2x+1)\sqrt{1-x^2}\,dx$

$= 2\int_0^1 (1+x^2)\sqrt{1-x^2}\,dx \xrightarrow{x=\sin t} 2\int_0^{\frac{\pi}{2}} (1+\sin^2 t)\cos^2 t\,dt = 2\int_0^{\frac{\pi}{2}} (1+\sin^2 t)(1-\sin^2 t)\,dt$

$= 2\int_0^{\frac{\pi}{2}} 1-\sin^4 t\,dt = 2\int_0^{\frac{\pi}{2}} dx - 2\int_0^{\frac{\pi}{2}} \sin^4 t\,dt$

$= \pi - 2I_4 = \pi - 2\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{5}{8}\pi$

例4. ① $f, g \in C[-a,a]$, $f(-x)+f(x)\equiv A$, $g(x)$ 为偶. 证 $\int_{-a}^a f(x)g(x)\,dx = A\int_0^a g(x)\,dx$

② 求 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{arctan} e^x \cdot \sin^2 x\,dx$

解: ① 左 $= \int_0^a [f(x)g(x)+f(x)g(-x)]\,dx$

$= \int_0^a Ag(x)\,dx = A\int_0^a g(x)\,dx$

② $\int_0^{\frac{\pi}{2}} (\operatorname{arctan} e^x + \operatorname{arctan} e^{-x})\sin^2 x\,dx$

$\because (\operatorname{arctan} e^x + \operatorname{arctan} e^{-x})' = \frac{e^x}{1+e^{2x}} - \frac{e^{-x}}{1+e^{-2x}} = \frac{e^x}{1+e^x} - \frac{e^x}{e^{2x}+1} = 0$

$\therefore \operatorname{arctan} e^x + \operatorname{arctan} e^{-x} \equiv A$ 取 $x=0$, $A=\frac{\pi}{2}$

原式 $= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{\pi}{2}\times I_2 = \frac{\pi}{2}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{\pi^2}{8}$.

② $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

Cps: $\int_a^{a+b} \xrightarrow{x-a=t} \int_a^b$).

左 $= \int_0^{\frac{\pi}{2}} f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx$

而 $\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx \xrightarrow{x-\frac{\pi}{2}=t} \int_0^{\frac{\pi}{2}} f(\cos t)\,dt = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

∴ 左 = 右.

例 6. $I = \int_{-\pi}^{\pi} \frac{\sin^2 x}{1+e^x}\,dx$

解: $I = \int_0^{\pi} \left(\frac{\sin^2 x}{1+e^x} + \frac{\sin^2 x}{1+e^{-x}}\right) dx$

$= \int_0^{\pi} \left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right) \sin^2 x\,dx$

$= \int_0^{\pi} \left(\frac{1}{1+e^x} + \frac{e^x}{1+e^x}\right) \sin^2 x\,dx = \int_0^{\pi} \sin^2 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = 2I_2$

$\Rightarrow 2I_2 = 2 \times \frac{\pi}{2} \times \frac{1}{2} = \frac{\pi}{2}$.

Note: $\int_0^{\pi} f(\cos x)\,dx$ 无上述性质. 若 $|\cos x|$, $\cos^{2n} x$ 可以用.

③ $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

例 5. $I = \int_0^{\pi} x \sin^2 x\,dx$

解: $I = \frac{\pi}{2}\int_0^{\pi} \sin^2 x\,dx = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \pi I_2 = \frac{\pi^2}{4}$.

例 6 $I=\int_0^{\pi} x\cdot\frac{\sin x}{1+\cos^2 x}\,dx$

解: $I=-\frac{\pi}{2}\cdot\int_0^{\pi}\frac{d\cos x}{1+\cos^2 x}=-\frac{\pi}{2}\arctan\cos x\Big|_0^{\pi}=-\frac{\pi}{2}\cdot\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=\frac{\pi^2}{4}$

证明上述定理.

记 $I=\int_0^{\pi} x f(\sin x)\,dx \xrightarrow{x=\pi-t} \int_{\pi}^{0}(\pi-t)f(\sin t)(-dt)$

$=\int_0^{\pi}(\pi-t)f(\sin t)\,dt=\int_0^{\pi}(\pi-x)f(\sin x)\,dx$

$=\pi\int_0^{\pi}f(\sin x)\,dx-I$

$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)\,dx$

3. $f(x)$ 连续, 以 T 为周期

① $\int_a^{a+T}f(x)\,dx=\int_0^{T}f(x)\,dx$.

② $\int_0^{nT}f(x)\,dx=n\int_0^{T}f(x)\,dx$.

第三节 广义积分（反常积分）

正常积分 {
- 积分区间有限
- 被积函数有限区间上连续，或$\exists$有限第一类间断点
}

一. 区间无限的反常积分

(一) $f(x) \in C[a,b]$, $\int_a^{+\infty} f(x)dx$

def 一. $\int_a^b f(x)dx = F(b) - F(a)$

① $F(b)-F(a)$ 与 $\int_a^{+\infty} f(x)dx$ 不同.

② $\lim\limits_{b\to+\infty}[F(b)-F(a)]$ 与 $\int_a^{+\infty} f(x)dx$ 相同.

If $\lim\limits_{b\to+\infty}[F(b)-F(a)] = A$. 则 $\int_a^{+\infty} f(x)dx = A$

If $\lim\limits_{b\to+\infty}[F(b)-F(a)]$ 不$\exists$，则 发散.

例 1. $\int_1^{+\infty} \frac{dx}{1+x^2}$

解: $\int_1^b \frac{dx}{1+x^2} = \operatorname{arctan} b - \frac{\pi}{4}$

$\therefore \lim\limits_{b\to+\infty} \operatorname{arctan} b - \frac{\pi}{4} = \frac{\pi}{2}$ 即 原式 $= \frac{\pi}{2}$.

判别法.

① If $\exists \alpha > 1$; $\lim\limits_{x\to+\infty} x^\alpha f(x)$ $\exists$. 收敛

② If $\exists \alpha \leq 1$, $\lim\limits_{x\to+\infty} x^\alpha f(x) = k (\neq 0)$ 或 ∞. 发散.

例2. 判断 $\int_0^{+\infty} \frac{\sqrt{x}}{1+x^2}dx$ 的敛散性.

解: $\because \lim_{x\to+\infty} x^{\frac{3}{2}} \cdot \frac{\sqrt{x}}{1+x^2} = 1$ 且 $\alpha = \frac{3}{2} > 0$

$\therefore$ 收敛.

☆Note. Γ-函数.

$$\Gamma(\alpha) \triangleq \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$$

① $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

② $\Gamma(n+1) = n!$

③ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

例3. ① $\int_0^{+\infty} x^5 e^{-x} dx$

解: 原式 $= \Gamma(6) = 5! = 120$

② $\int_0^{+\infty} x\sqrt{x} e^{-x} dx$

解: 原式 $= \Gamma(\frac{5}{2}) = \Gamma(\frac{3}{2}+1) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\Gamma(\frac{1}{2}+1) = \frac{3}{2}\times\frac{1}{2}\Gamma(\frac{1}{2})$

$= \frac{3}{4}\sqrt{\pi}$.

例4 $\int_0^{+\infty} x^2 e^{-x^2} dx \xrightarrow{x^2=t} \int_0^{+\infty} t e^{-t} \frac{1}{2\sqrt{t}} dt$

$= \frac{1}{2}\int_0^{+\infty} \sqrt{t} e^{-t} dt = \frac{1}{2}\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}+1) = \frac{1}{2}\times\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{4}\sqrt{\pi}$.

(二) $f(x) \in (-\infty, a]$, $\int_{-\infty}^{a} f(x)dx$

def - $\int_{b}^{a} f(x)dx = F(a) - F(b)$

If $\lim_{b\to-\infty} F(a) - F(b) = A$, 则 $\int_{-\infty}^{a} f(x)dx = A$

If $\lim_{b\to-\infty} F(a) - F(b)$ 不∃, 则发散.

判别法

If $\exists \alpha > 1$, $\lim_{x\to-\infty} x^{\alpha} f(x)$ ∃ $\Rightarrow$ 收敛.

If $\exists \alpha \le 1$, $\lim_{x\to-\infty} x^{\alpha} f(x) = k\ (\neq 0$ 或 $\infty)$ $\Rightarrow$ 发散.

二. 区间有限

(一) $f(x) \in C(a, b]$ 且 $f(a+0) = \infty$

def - $\forall \varepsilon > 0$, $\int_{a+\varepsilon}^{b} f(x)dx = F(b) - F(a+\varepsilon)$

① $F(b) - F(a+\varepsilon)$ 与 $\int_{a}^{b} f(x)dx$ 不同.

② $\lim_{\varepsilon\to0^+}[F(b) - F(a+\varepsilon)]$ 与 $\int_{a}^{b} f(x)dx$ 同.

If $\lim_{\varepsilon\to0^+}[F(b) - F(a+\varepsilon)] = A$, $\int_{a}^{b} f(x)dx = A$

If $\lim_{\varepsilon\to0^+}[F(b) - F(a+\varepsilon)]$ 不∃. 发散.

例 4. $\int_1^2 \frac{dx}{x\sqrt{x-1}}$

解: $\forall \varepsilon>0$ $\int_{1+\varepsilon}^2 \frac{dx}{x\sqrt{x-1}} = 2\int_{1+\varepsilon}^2 \frac{dx}{x\cdot 2\sqrt{x-1}} = 2\int_{1+\varepsilon}^2 \frac{d\sqrt{x-1}}{x}$

$\xrightarrow{t=\sqrt{x-1}} 2\int_{\sqrt{\varepsilon}}^1 \frac{dt}{t^2+1} = 2\arctan t\Big|_{\sqrt{\varepsilon}}^1 = \frac{\pi}{2} - 2\arctan\sqrt{\varepsilon}$

即 $\lim_{\varepsilon\to 0^+} \frac{\pi}{2} - 2\arctan\sqrt{\varepsilon} = \frac{\pi}{2}$.

$\therefore \int_1^2 \frac{dx}{x\sqrt{x-1}} = \pi/2$.

例 5. $\int_1^2 \frac{dx}{(x-1)^2}$

解: $\forall \varepsilon>0$, $\int_{1+\varepsilon}^2 \frac{1}{(x-1)^2}dx = -(x-1)^{-1}\Big|_{1+\varepsilon}^2 = -1+\frac{1}{\varepsilon}$

$\lim_{\varepsilon\to 0^+} -1+\frac{1}{\varepsilon} = +\infty$ $\therefore$ 原式发散

判别法

① $\exists \lambda<1$: 使 $\lim_{x\to a^+} (x-a)^\lambda f(x)$ 存. 收敛

② $\exists \lambda\geq 1$: $\lim_{x\to a^+} (x-a)^\lambda f(x) = k\ (\neq 0)$ 或 ∞. 发散.

例 6. $\int_0^1 \frac{dx}{\sqrt{x+x}}$

解: $\because \lim_{x\to 0^+} (x-0)^{\frac{1}{2}} \frac{1}{\sqrt{x}\cdot\sqrt{x+1}} = 1$ 且 $\lambda = \frac{1}{2} < 1$

$\therefore$ 收敛

$\int_0^1 \frac{dx}{\sqrt{x+x}} = 2\int_0^1 \frac{dx}{2\sqrt{x}\sqrt{x+1}} = 2\int_0^1 \frac{d\sqrt{x}}{\sqrt{x+1}} \xrightarrow{\sqrt{x}=t} 2\int_0^1 \frac{dt}{\sqrt{1+t^2}}$

$= 2\ln(x+\sqrt{x+1})\Big|_0^1 = 2\ln(1+\sqrt{2})$

(二) $f(x) \in C[a,b)$，且 $f(b-0)=\infty$

def - $\forall \varepsilon>0$ $\int_a^{b-\varepsilon} f(x)dx = F(b-\varepsilon)-F(a)$

① $F(b-\varepsilon)-F(a)$ 与 $\int_a^b f(x)dx$ 不同　② $\lim_{\varepsilon\to 0^+} F(b-\varepsilon)-F(a)$ 与 $\int_a^b f(x)dx$ 同.

If $\lim_{\varepsilon\to 0^+}(F(b-\varepsilon)-F(a)) = A$，则 $\int_a^b f(x)dx = A$.

If $\lim_{\varepsilon\to 0^+}[F(b-\varepsilon)-F(a)]$ 不∃，则 发散.

判别法.

① If $\exists \alpha<1$，使 $\lim_{x\to b}(b-x)^\alpha f(x)$ ∃，收敛

② If $\exists \alpha\geq 1$，$\lim_{x\to b}(b-x)^\alpha f(x) = k$ ($k\neq 0$ 或 ∞)，发散.

例 7. $\int_0^1 \ln(1-x)\,dx$.

解: $\forall \varepsilon>0$, $\int_0^{1-\varepsilon} \ln(1-x)\,dx$

$$\Rightarrow x\ln(1-x)\Big|_0^{1-\varepsilon} + \int_0^{1-\varepsilon} \frac{x}{1-x}dx$$

$$\Rightarrow x\ln(1-x)\Big|_0^{1-\varepsilon} + \int_0^{1-\varepsilon} -1+\frac{1}{1-x}dx$$

$$\Rightarrow x\ln(1-x)\Big|_0^{1-\varepsilon} - (1-\varepsilon) - \ln|1-x|\Big|_0^{1-\varepsilon}$$

$$\Rightarrow (1-\varepsilon)\ln\varepsilon - (1-\varepsilon) - \ln\varepsilon = -\varepsilon\ln\varepsilon - 1 + \varepsilon$$

$\lim_{\varepsilon\to 0^+}$ $\lim_{\varepsilon\to 0^+} \varepsilon\ln\varepsilon = \lim_{\varepsilon\to 0^+} \frac{\ln\varepsilon}{\frac{1}{\varepsilon}} = 0$

$\therefore \lim_{\varepsilon\to 0^+} \varepsilon\ln\varepsilon - 1 + \varepsilon = -1$　即收敛，原式 $=-1$.

例8. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

解: $\lim_{x\to 1^-} (1-x)^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{2}}$ 且 $\alpha < 1$

∴收敛

原式 $= \arcsin x \Big|_0^1 = \pi/2$.

第四节 定积分应用

一. 几何应用

(一) 面积

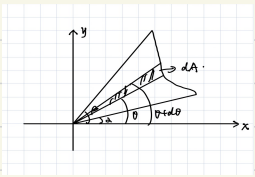
1. $A=\int_a^b f(x)\,dx$

2. $L: r=r(\theta)\quad(\alpha\le\theta\le\beta)$

   1° 取 $[\theta,\theta+d\theta]\subset[\alpha,\beta]$

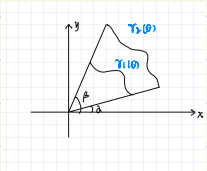
   2° $dA=\frac{1}{2}r^2(\theta)\,d\theta$

   3° $A=\frac{1}{2}\int_\alpha^\beta r^2(\theta)\,d\theta$



3.

$$A=\frac{1}{2}\int_\alpha^\beta [r_2^2(\theta)-r_1^2(\theta)]\,d\theta.$$



4.

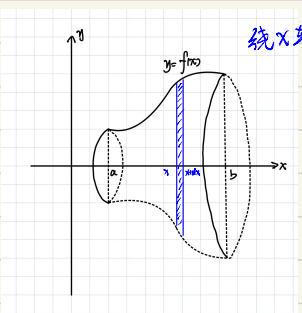
1° 取 $[x,x+dx]\subset[a,b]$

2° $dA=2\pi f(x)\,ds$

$=2\pi f(x)\sqrt{1+f'^2(x)}\,dx$

$=2\pi y\sqrt{1+y'^2}\,dx$

3 $S=2\pi\int_a^b y\sqrt{1+y'^2}\,dx.$

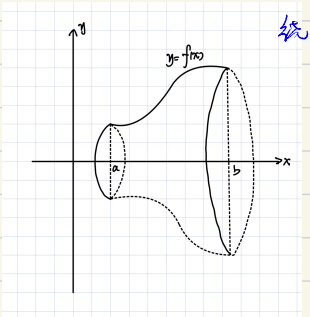


(二) 体积.

1. 1° 取 $[x, x+dx] \subset [a,b]$

2° $dV = \pi f^2(x) \cdot dx$

3° $V_x = \pi \int_a^b f^2(x)\, dx$



绕 x

2. 1° 取 $[x, x+dx] \subset [a,b]$

2° $dV = 2\pi x \cdot y\, dx$

$= 2\pi x f(x)\, dx$

3° $V_y = 2\pi \int_a^b x f(x)\, dx$

绕 y 轴.

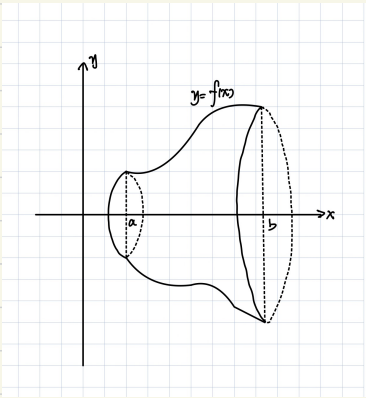
3.

上下不规则

1° 取 $[x, x+dx] \in [a,b]$

2° $dV = A(x) \cdot dx$;

3° $V = \int_a^b A(x)\, dx$.



(三) 弧长.

1. $L: y=f(x) \quad (a\le x\le b)$

1° 取 $[x, x+dx] \subset [a,b]$

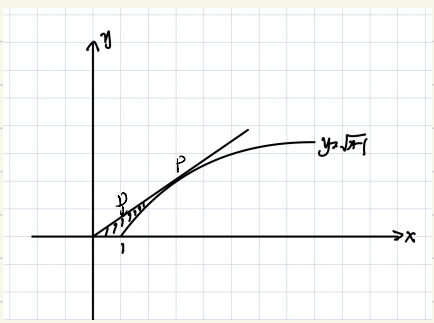
2° $ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{1+f'^2(x)}\,dx$

3° $S = \int_a^b \sqrt{1+f'^2(x)}\,dx$.

2. $L: \begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad (\alpha\le t\le\beta)$

$S = \int_\alpha^\beta \sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$.

例 1.



① 求切线方程.

解: 设 $P(a, \sqrt{a-1})$ $\therefore k = \frac{\sqrt{a-1}}{a}$

而 $\frac{1}{2\sqrt{a-1}} = \frac{\sqrt{a-1}}{a} \Rightarrow a = 2(a-1)$ $\therefore a=2$

$\therefore y = \frac{1}{2}x$

切线: $y=\frac{1}{2}x$

② 求 D 的面积.

解: $A = \frac{1}{2}\times 2\times 1 - \int_1^2 \sqrt{x-1}\,dx$

$= 1 - \frac{2}{3}(x-1)^{\frac{3}{2}}\Big|_1^2$

$= \frac{1}{3}$

③ 使 D 绕 X 轴一周表面积.

解: $S_{外}$:

1° 取 $[x, x+dx] \subset [0,2]$.

2° $dS = 2\pi y \cdot ds = 2\pi y\sqrt{1+y'^2}\,dx$

3° $S = \int_0^2 2\pi(\frac{1}{2}x)\sqrt{\frac{5}{4}}\,dx$

$= \frac{\sqrt{5}}{2}\pi \int_0^2 x\,dx = \sqrt{5}\pi$.

$S_{内}$:

$ds = 2\pi y\sqrt{1+y'^2}\,dx$

$= 2\pi\sqrt{x-1}\sqrt{1+\frac{1}{4(x-1)}}\,dx$

$= 2\pi\sqrt{x-1}\,\frac{\sqrt{4x-3}}{2\sqrt{x-1}}\,dx$

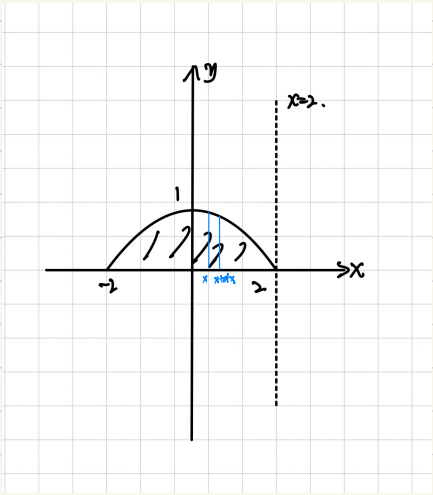
$= \pi\sqrt{4x-3}\,dx$

$S = \pi\int_1^2 \sqrt{4x-3}\,dx = \frac{\pi}{4}\cdot\frac{2}{3}(4x-3)^{\frac{3}{2}}\Big|_1^2 = \frac{7\pi}{6}(5\sqrt{5}-1)$

$\therefore S_{表} = S_{内} + S_{外} = \frac{11\sqrt{5}-1}{6}\pi$

例 2.

已知半椭圆如图，绕 $x=2$ 旋转，求其体积。



1° 取 $[x, x+dx] \subset [-2, 2]$

2° $dV = 2\pi(2-x) \cdot y \cdot dx$

$\because \frac{x^2}{4}+y^2=1 \quad \therefore y=\sqrt{1-\frac{x^2}{4}}$

$\therefore dV = 2\pi(2-x)\frac{1}{2}\sqrt{4-x^2}\,dx$

$= \pi(2-x)\sqrt{4-x^2}\,dx$

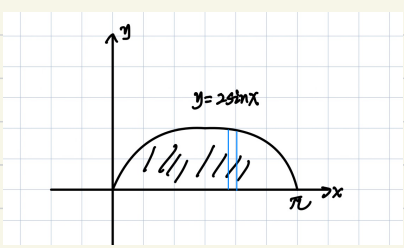
3° $V = \pi\int_{-2}^{2}(2-x)\sqrt{4-x^2}\,dx$

$= 4\pi\int_{0}^{2}\sqrt{4-x^2}\,dx \xrightarrow{x=2\sin t} 4\pi\int_{0}^{\frac{\pi}{2}} 2\cos t \cdot 2\cos t\,dt$

$= 16\pi \times I_2 = 16\pi \cdot \frac{1}{2}\times\frac{\pi}{2} = 4\pi^2$

例 3

求 V_y



解：1° 取 $[x, x+dx] \subset [0, \pi]$

2° $dV = 2\pi x \cdot y \cdot dx$

$= 4\pi x\sin x\,dx$

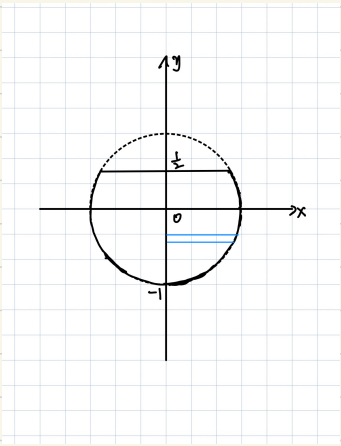
3° $V = 4\pi\int_{0}^{\pi} x\sin x\,dx$

$= 4\pi\frac{\pi}{2}\int_{0}^{\pi}\sin x\,dx$

$= 2\pi^2 \cdot 2I_1 = 4\pi^2$

4.

求 V_y



解：1° 取 $[y, y+dy] \subset [-1, \frac{1}{2}]$

2° $dV = \pi x^2 \cdot dy$

$\because x^2+y^2=1 \quad \therefore x^2 = 1-y^2$

$\therefore dV = \pi(1-y^2)\,dy$

3° $V = \pi \int_{-1}^{\frac{1}{2}} (1-y^2)\,dy$

$= \pi\left(y\Big|_{-1}^{\frac{1}{2}} - \frac{1}{3}y^3\Big|_{-1}^{\frac{1}{2}}\right)$

$= \pi \cdot \left(\frac{3}{2} - \frac{3}{8}\right) = \frac{9}{8}\pi.$

第六章 多元函数微分学

第一节 基本概念

1. 极限

一元：If $\forall \varepsilon>0$，$\exists \delta>0$，当 $0<|x-x_0|<\delta$ 时，$|f(x)-A|<\varepsilon$

$\lim\limits_{x\to x_0} f(x)=A$ 且 $\lim\limits_{x\to x_0} f(x)\exists \Leftrightarrow f(x_0-0)$ 与 $f(x_0+0)\exists$ 且等.

二元：$f(x,y)$ 在 $M_0(x_0,y_0)$ 去心邻域内有定义，

If $\forall \varepsilon>0$，$\exists \delta>0$，当 $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ 时，

$|f(x,y)-A|<\varepsilon \Rightarrow \lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=A$.

例1. $\lim\limits_{\substack{x\to 0\\ y\to 2}} \dfrac{\sqrt{1+xy}-\sqrt{xy}}{x}$

原式 $=\lim\limits_{y\to 0} y\cdot \dfrac{\sqrt{1+xy}-\sqrt{xy}}{xy} \overset{xy=t}{=\!=} 2\lim\limits_{t\to 0} \dfrac{\sqrt{1+t}-\sqrt{t}}{t}$

$=2\lim\limits_{t\to 0} \dfrac{1}{\sqrt{1+t}+\sqrt{t}}\cdot \dfrac{t}{t}=4\times\dfrac{1}{2}=2$.

例2. $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$ $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$?

解：$\lim\limits_{\substack{x\to 0\\ y=x}} f(x,y)=\dfrac{1}{2}$

$\lim\limits_{\substack{x\to 0\\ y=-x}} f(x,y)=-\dfrac{1}{2}$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不$\exists$

2. 连续

一元 - 若 $\lim\limits_{x\to x_0} f(x)=f(x_0) \Leftrightarrow f(x_0-0)=f(x_0+0)=f(x_0)$.

二元 - 若 $\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)=f(x_0,y_0)$，则 $f(x,y)$ 在 (x_0,y_0) 连续.

3. 偏导数 - $z=f(x,y)$ $((x,y)\in D)$，$M_0(x_0,y_0)\in D$

$\Delta z_x = f(x_0+\Delta x, y_0)-f(x_0,y_0)$ Δz_x — $f(x,y)$ 在 M_0 处关于 x 的偏增量.

$\Delta z_y = f(x_0, y_0+\Delta y)-f(x_0,y_0)$

$\Delta z = f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$ — $f(x,y)$ 在 M_0 处的全增量.

若 $\lim\limits_{\Delta x\to 0} \dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$ $\left(=\lim\limits_{x\to x_0}\dfrac{f(x,y_0)-f(x_0,y_0)}{x-x_0}\right)$ 存.

称 $f(x,y)$ 在 M_0 关于 x 可偏导，记 $f_x(x_0,y_0)$，$\left.\dfrac{\partial z}{\partial x}\right|_{M_0}$

4. 可全微

一元 - $y=f(x)$，$(x\in D)$ $x_0\in D$，$\Delta y=f(x_0+\Delta x)-f(x_0)$

若 $\Delta y = \underline{A\Delta x}+o(\Delta x)$，则 $y=f(x)$ 在 x_0 处可微

↳ $dy|_{x=x_0}$，$A\Delta x$

① 可导 $\Leftrightarrow$ 可微

② $A=f'(x_0)$

③ $y=f(x)$ 可导 $\Rightarrow$ $dy=f'(x)dx$.

二元. $z=f(x,y)$ $((x,y)\in D,\ M_0(x_0,y_0)\in D)$

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ If $\Delta z = A\Delta x + B\Delta y + O(\rho)$ $(\rho=\sqrt{\Delta x^2+\Delta y^2})$

则称 $z=f(x,y)$ 在 M_0 处可全微. 其中 $A\Delta x+B\Delta y \triangleq dz\big|_{M_0}$

$\Rightarrow A\Delta x+B\Delta y \to f(x,y)$ 在 M_0 处全微分.

第二节 多元函数理论

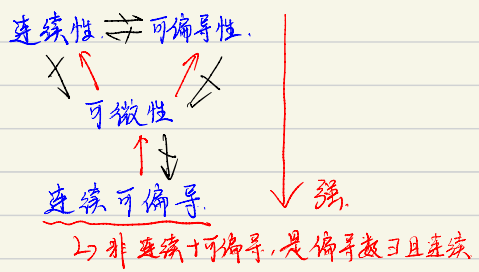
一. $f(x,y)\in C(D)$, D-有界闭区域, 性质:

① $\exists\ m, M$

② $\exists\ k>0$, 使 $|f(x)|\le k$.

③ $\forall \delta\in[m,M]$, 则 $\exists (\xi,\eta)\in D$, $f(\xi,\eta)=\delta$.

二. 二元函数. 连续性, 可偏导性, 可微性.



⇒ 非连续+可偏导, 是偏导数∃且连续.

证: ① "可微" $\Rightarrow$ "连续"

$\Delta z = f(x,y)-f(x_0,y_0) = A(x-x_0)+B(y-y_0)+O(\sqrt{(x-x_0)^2+(y-y_0)^2})$

$\because \lim_{\substack{x\to x_0\\ y\to y_0}} \Delta z = 0 \quad \therefore \lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f(x_0,y_0)$

② "可微" $\Rightarrow$ "可偏导"

$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = A\Delta x + B\Delta y + o(\rho)$

取 $\Delta y = 0 \Rightarrow \Delta z_x = A\Delta x + o(\Delta x) \Rightarrow \frac{\Delta z}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x} \Rightarrow \lim \frac{\Delta z}{\Delta x} = A$

同理 $\frac{\Delta z}{\Delta y} = B$.

反例1. $f(x,y) = \sqrt{x^2+y^2}$ 在 $(0,0)$

$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0) \Rightarrow$ 连续

$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{|x|}{x}$ 不存在 $\Rightarrow$ 对 x,y 不可偏导.

反例2. $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

$\lim_{\substack{x\to 0\\ y=x}} f(x,y) = \frac{1}{2}$ $\lim_{\substack{x\to 0\\ y=-x}} f(x,y) = -\frac{1}{2} \Rightarrow$ 不连续

$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$ $\lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$

即 $f_x(0,0) = f_y(0,0) = 0$ $\Rightarrow$ 可偏导.

三. 设 $z=f(x,y)$ 二阶连续可偏导 $\Rightarrow \frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}$.

求偏导类型

(一) 显函数求偏导. $z=f(x,y)$.

例1. $z=\ln^2 \operatorname{arccot}\frac{x+y}{x-y}$

$$\frac{\partial z}{\partial x}=2\ln \operatorname{arccot}\frac{x+y}{x-y}\cdot\frac{1}{\operatorname{arccot}\frac{x+y}{x-y}}\cdot\frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}\cdot\frac{(x-y)-(x+y)}{(x-y)^2}$$

例2. $z=(1+x^2+y^2)^{xy}$

$$z=e^{xy\ln(1+x^2+y^2)} \quad \therefore \frac{\partial z}{\partial x}=e^{xy\ln(1+x^2+y^2)}\cdot\left[y\ln(1+x^2+y^2)+xy\cdot\frac{2x}{1+x^2+y^2}\right]$$

(二) 复合函数求偏导.

① $z=f(x^2+y^2)$. $z=f(u)$. $u=x^2+y^2$

② $z=f(e^x,e^y)$ $z=f(u,v)$ $\begin{cases}u=e^x\\ v=e^y\end{cases}$

$z=f(u,v)$. $\frac{\partial z}{\partial u}=f_u(u,v)=f_1(u,v)=f_1$

$\frac{\partial z}{\partial v}=f_v(u,v)=f_2(u,v)=f_2$.

$\frac{\partial^2 z}{\partial u^2}=f_{11}$. $\frac{\partial^2 z}{\partial u \partial v}=f_{12}$.

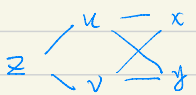
例 $z=e^{u+\sin v}$，$u=xy$，$v=\frac{x}{y}$，求 $\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

解：法一. $z=e^{xy+\sin\frac{x}{y}}$

$$\frac{\partial z}{\partial x}=e^{xy+\sin\frac{x}{y}}\cdot\left(2xy^2-\frac{y}{x}\cos\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial y}=e^{xy+\sin\frac{x}{y}}\left(2yx^2+\frac{1}{x}\cos\frac{x}{y}\right)$$

法二.



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^{u+\sin v}\cdot 2u\cdot y+e^{u+\sin v}\cdot\cos v\cdot\left(-\frac{y}{x^2}\right)$$

$$=e^{xy+\sin\frac{x}{y}}\left(2xy^2-\frac{y}{x^2}\cos\frac{x}{y}\right)$$

例2. $z=f(t^2,e^t)$，其中 $f(u,v)$ 二阶连续可偏导，求 $\frac{d^2z}{dt^2}$.

解：$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}$

$$=f_1\cdot 2t+e^tf_2$$

$$\frac{d^2z}{dt^2}=2f_1+2t\cdot(f_{11}2t+e^tf_{12})+e^tf_2+e^t(f_{21}2t+f_{22}e^t)$$

$$=2f_1+e^tf_2+4t^2f_{11}+4te^tf_{12}+e^tf_{22}$$

例3. $z=f(x+y^2,e^x\sin y)$，$f(u,v)$ 二阶连续可偏导，求 $\frac{\partial^2z}{\partial x\partial y}$

解：$\frac{\partial z}{\partial x}=f_1(x)'+f_2e^x\sin y$

$$\frac{\partial^2z}{\partial x\partial y}=2x[f_{11}2y+f_{12}e^x(\cos y)]+e^x\cos yf_2+e^x\sin y[f_{21}2y+f_{22}e^x\cos y]$$

$$=4xyf_{11}+(2xe^x\cos y+2ye^x\cos y)f_{12}+e^x\cos yf_2+e^{2x}\cos y\sin yf_{22}$$

例 $z=f(xy,x+y,x^2)$，则 f 二阶连续可偏导，求 $\frac{\partial^2 z}{\partial x\partial y}$

解：$\frac{\partial z}{\partial x}=f_1+yf_2+2xf_3$

$\frac{\partial^2 z}{\partial x\partial y}=(f_{11}+f_{12}x)+f_2+x(f_{21}+f_{22}x)+2x(f_{31}+xf_{32})$

$=f_{11}+(x+y)f_{12}+f_2+xyf_{22}+2xf_{31}+2x^2f_{32}$

3. 隐函数求偏导

① $F(x,y)=0 \Rightarrow$ 一个一元

② $F(x,y,z)=0 \Rightarrow$ 一个二元

③ $\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases} \Rightarrow$ 二个一元

④ $\begin{cases}F(x,y,u,v)=0\\G(x,y,u,v)=0\end{cases} \Rightarrow$ 二个二元

例 $\begin{cases}x+2y-z=3\\x^2+4y^2-z^2=21\end{cases} \Rightarrow$ 求 $\frac{dz}{dx}$

解：1° $\begin{cases}x+2y-z=3\\x^2+4y^2-z^2=21\end{cases} \Rightarrow \begin{cases}y=y(x)\\z=z(x)\end{cases}$

2° $\begin{cases}1+2\frac{dy}{dx}-\frac{dz}{dx}=0\\2x+8y\cdot\frac{dy}{dx}-2z\frac{dz}{dx}=0\end{cases} \Rightarrow \frac{dz}{dx}=\frac{x+4y}{z-2y}$

例2. $y=f(x,t)$. $F(x^2,\sin y,e^t)=0$. f,F 连续可偏导.

求 $\frac{dt}{dx}$.

解: 1° $\begin{cases} f(x,t)=y \\ F(x^2,\sin y,e^t)=0 \end{cases} \Rightarrow \begin{cases} y=y(x) \\ t=t(x) \end{cases}$

2° $\begin{cases} \frac{dy}{dx}=f_1+f_2\frac{dt}{dx} \\ F_1\cdot 2x+F_2\cos y\cdot\frac{dy}{dx}+F_3e^t\cdot\frac{dt}{dx}=0 \end{cases} \Rightarrow \frac{dt}{dx}=\cdots$

例3. $\begin{cases} xu-yv=1 \\ u^2+yv=x^2+yu \end{cases}$. 求 $\frac{\partial u}{\partial x}$ $\frac{\partial v}{\partial x}$

解: 1. $\begin{cases} xu-yv=1 \\ u^2+yv=x^2+yu \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

2° $\begin{cases} x\frac{\partial u}{\partial x}+u-y\frac{\partial v}{\partial x}=0 \\ 2u\cdot\frac{\partial u}{\partial x}+y\cdot\frac{\partial y}{\partial x}-2x-y\frac{\partial u}{\partial x}=0 \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x}=\cdots \\ \frac{\partial v}{\partial x}=\cdots \end{cases}$

第三节 多元函数微分学求数应用

一元：$y=f(x)$ ① $x\in D$； ② $f(x)$ $\begin{cases}=0\\ \text{不}\exists\end{cases}$ ③ 判别法

Th1（第一充分…）

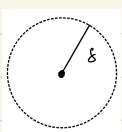
Th2（第二充分…） $f'(x_0)=0$ $f''(x_0)$ $\begin{cases}>0 \text{ 小}\\ <0 \text{ 大}\end{cases}$

二元

- def. - $z=f(x,y)$，$(x,y)\in D$，$M_0(x_0,y_0)\in D$

If $\exists\delta>0$，当 $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$ 时，有 $f(x,y)<f(x_0,y_0)$ 极大

$f(x,y)>f(x_0,y_0)$ 极小



二. 极值问题

(一) 二元函数无条件极值

$z=f(x,y)$ $(x,y)\in D$ D一定是开区域

1° $\begin{cases}\frac{\partial z}{\partial x}=\cdots=0\\ \frac{\partial z}{\partial y}=\cdots=0\end{cases}\Rightarrow\begin{cases}x=?\\ y=?\end{cases}$

2° 设 (x_0,y_0) 为一个驻点，

$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(x_0,y_0)}$ $B=\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{(x_0,y_0)}$ $C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(x_0,y_0)}$

$AC-B^2$ $\begin{cases}<0 & \text{非极值点}\\ >0 & \text{为极值点，}\begin{cases}A<0 & \text{极大}\\ A>0 & \text{极小}\end{cases}\end{cases}$

例1 $z=f(x,y)=x^3-3x^2-9x+y^2+2y-2$，求极值

解 1° 由 $\begin{cases} \frac{\partial z}{\partial x}=3x^2-6x-9=0 \\ \frac{\partial z}{\partial y}=2y+2=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=-1 \end{cases}; \begin{cases} x=3 \\ y=-1 \end{cases}$

2° $\frac{\partial^2 z}{\partial x^2}=6x-6$ $\frac{\partial^2 z}{\partial y^2}=2$

$\frac{\partial^2 z}{\partial x \partial y}=0$

当 $(x,y)=(-1,-1)$ 时 $A=-12$, $B=0$, $C=2$ $AC-B^2<0$ 非

$(x,y)=(3,-1)$ 时 $A=12$, $B=0$, $C=2$ $AC-B^2>0$ 是，$A>0$，极小

∴ 极小值 $\Rightarrow f(3,-1)=-27+1-2-2=-30$

(二) 多元函数条件极值 — Lagrange 乘子法

Case 1 $z=f(x,y)$ s.t. $\varphi(x,y)=0$ (等式)

1° $F=f(x,y)+\lambda\varphi(x,y)$

2° 由 $\begin{cases} F_x=f_x+\lambda\varphi_x=0 \\ F_y=f_y+\lambda\varphi_y=0 \\ F_\lambda=\varphi(x,y)=0 \end{cases} \Rightarrow \begin{cases} x=? \\ y=? \end{cases}$

例1. $u=u(x,y)$ 满足 ① $du=2x\,dx-2y\,dy$; ② $u(0,0)=2$

求 (1) $u(x,y)$

(2) $u(x,y)$ 在 $\underbrace{x^2+4y^2\le 4}_{<\text{为无条件}\quad =\text{为条件}}$ 上的 m, M.

(1) 法一. $du=2x\,dx-2y\,dy \Rightarrow u=x^2-y^2+C$

而 $u(0,0)=2$ $\therefore$ $u=x^2-y^2+2$

法二.

$du=2x\,dx-2y\,dy$. $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$.

$\frac{\partial u}{\partial x}=2x \Rightarrow u=x^2+\boxed{\varphi(y)} \Rightarrow \frac{\partial u}{\partial y}=\varphi'(y)=-2y \Rightarrow \varphi(y)=-y^2+C$

$\therefore$ $u=x^2-y^2+C$.

(2) ① 当 $x^2+4y^2<4$ 时. 无条件

$$\begin{cases}\frac{\partial u}{\partial x}=2x=0\\ \frac{\partial u}{\partial y}=-2y=0\end{cases} \Rightarrow \begin{cases}x=0\\ y=0\end{cases} \qquad u(0,0)=2$$

② 当 $x^2+4y^2-4=0$ 时. 有条件.

设 λ.

$F=x^2-y^2+2+\lambda x^2+4\lambda y^2-4\lambda$

$$\begin{cases}F_x=2x+2\lambda x=0\\ F_y=-2y+8\lambda y=0\\ F_\lambda=x^2+4y^2-4\end{cases} \Rightarrow \begin{cases}x(\lambda+1)=0 & ①\\ y(4\lambda-1)=0 & ②\\ x^2+4y^2=4\end{cases} \Rightarrow ①,② \text{只看一个}$$

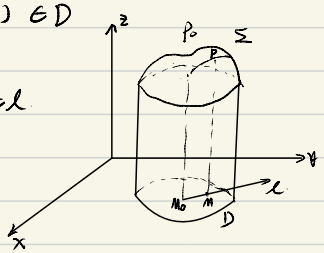
只看①为例. $\Rightarrow \begin{cases}x=0\\ y=\pm1\end{cases}$; $\begin{cases}x=\pm2\\ y=0\end{cases}$ $\quad u(0,\pm1)=1$, $u(\pm2,0)=6$

即 $m=1$, $M=6$

补充：方向导数与梯度

一. 方向导数

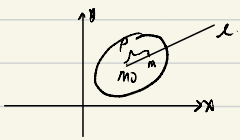
Σ: $z=f(x,y)$, $(x,y)\in D$, $M_0(x_0,y_0)\in D$



在 xOy 面内过 M_0 作射线 l，取 $M(x_0+\Delta x, y_0+\Delta y)\in l$

$\rho=\sqrt{\Delta x^2+\Delta y^2}$ $\quad \Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$

$\lim\limits_{\rho\to 0}\dfrac{\Delta z}{\rho}$ 存在.



(一) 定义

定义1. $z=f(x,y)$ $(x,y)\in D$. $M_0(x_0,y_0)\in D$.

在 xOy 面内过 M_0 作射线 l. 取 $M(x_0+\Delta x, y_0+\Delta y)\in l$.

$\rho=\sqrt{\Delta x^2+\Delta y^2}$ $\quad \Delta z=f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$

若 $\lim\limits_{\rho\to 0}\dfrac{\Delta z}{\rho}$ 存在. 称此极限为 $z=f(x,y)$ 在 M_0 处沿 l 的方向导数. 记 $\left.\dfrac{\partial z}{\partial l}\right|_{M_0}$

定义2. $u=f(x,y,z)$, $(x,y,z)\in\Omega$. $M_0(x_0,y_0,z_0)\in\Omega$.

过 M_0 作射线 l. 取 $M(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)\in l$.

$\rho=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ $\quad \Delta u=f(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)-f(x_0,y_0,z_0)$

若 $\lim\limits_{\rho\to 0}\dfrac{\Delta u}{\rho}$ 存在. 称此极限为 $u=f(x,y,z)$ 在 M_0 处沿射线 l 的方向导数.

记 $\left.\dfrac{\partial u}{\partial l}\right|_{M_0}$.

(二) 方向导数计算方法

Th1. $z=f(x,y)$ 在 $M_0(x_0,y_0)$ 可微. 在 xOy 面内过 M_0 作射线 l. 方向角为 α,β

则 $\left.\frac{\partial z}{\partial l}\right|_{M_0} = f_x'(x_0,y_0)\cos\alpha + f_y'(x_0,y_0)\cos\beta$.

证: $\left.\frac{\partial z}{\partial l}\right|_{M_0} = \lim_{\rho\to 0}\frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta)-f(x_0,y_0)}{\rho}$

$\because f(x,y)$ 在 M_0 处可微.

$\therefore \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + o(\rho)$

$\left.\frac{\partial z}{\partial l}\right|_{M_0} = \lim_{\rho\to 0}\frac{\Delta z}{\rho} = f_x'(x_0,y_0)\cos\alpha + f_y'(x_0,y_0)\cos\beta$

Th2. $u=f(x,y,z)$ 在 $M_0(x_0,y_0,z_0)$ 可微. 过 M_0 引 l. l 方向角 α,β,γ

则 $\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0}\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0}\cos\beta + \left.\frac{\partial u}{\partial z}\right|_{M_0}\cos\gamma$

例1. 求 $z=xe^{2y}$ 在 $M_0(1,0)$ 沿从 M_0 到 $M(2,-1)$ 的方向导数.

$\left.\frac{\partial z}{\partial x}\right|_{M_0} = e^{2y}\big|_{M_0} = 1 \qquad \left.\frac{\partial z}{\partial y}\right|_{M_0} = 2xe^{2y}\big|_{M_0} = 2$

$\overrightarrow{M_0M}(1,-1) \quad \cos\alpha = \frac{1}{\sqrt{2}} \quad \cos\beta = \frac{-1}{\sqrt{2}}$

$\therefore \left.\frac{\partial z}{\partial l}\right|_{M_0} = -\frac{\sqrt{2}}{2}$

例2. $u=xy+yz+zx$ 在 $M_0(1,1,2)$ 沿 l 的方向导数，其中 l 与 x 轴、y 轴、z 轴夹角为 $\frac{\pi}{3}$、$\frac{\pi}{3}$、$\frac{\pi}{4}$

1° $\frac{\partial u}{\partial x}=y+z=3$　　$\frac{\partial u}{\partial y}=x+z=3$　　$\frac{\partial u}{\partial z}=x+y=2$

2° $\left.\frac{\partial u}{\partial l}\right|_{M_0}=3\times\frac{1}{2}+3\times\frac{1}{2}+2\times\frac{\sqrt{2}}{2}=\frac{3+2\sqrt{2}}{2}$

二. 梯度

$u=f(x,y,z)$　$M_0(x_0,y_0,z_0)\in\Omega$. 过 M_0，l 方向角 α,β,γ

$$\left.\frac{\partial u}{\partial l}\right|_{M_0}=\left.\frac{\partial u}{\partial x}\right|_{M_0}\cos\alpha+\left.\frac{\partial u}{\partial y}\right|_{M_0}\cos\beta+\left.\frac{\partial u}{\partial z}\right|_{M_0}\cos\gamma$$

$$=\underbrace{\left\{\left.\frac{\partial u}{\partial x}\right|_{M_0},\left.\frac{\partial u}{\partial y}\right|_{M_0},\left.\frac{\partial u}{\partial z}\right|_{M_0}\right\}}_{\text{常向量，与 } l \text{ 无关}}\cdot\underbrace{\{\cos\alpha,\cos\beta,\cos\gamma\}}_{\text{单位向量，与 } l \text{ 同向}}$$

$$=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}\Big|_{M_0}\cos\theta$$　当 $\theta=0$ 时，$\left.\frac{\partial u}{\partial l}\right|_{M_0}$ 取最大值

梯度 - $u=f(x_0,y_0,z)$ 在 M_0 梯度

$$\operatorname{grad}(u)\big|_{M_0}=\left\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right\}\Big|_{M_0}$$

一般 $u=f(x,y,z)$，$\operatorname{grad}u$ 为 $\left\{\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right\}$

梯度方向即函数增长速度 max 方向 / 方向导数取 Max 方向

第七章 微分方程

一. 基本概念

1. 微分方程 - 含有导数或微分的方程

2. 阶 : 微分方程中导数/微分的最高阶数

3 微分方程的解 — 使方程成立的函数

设 $u_1(x)$, $u_2(x)$ 为两个函数，且 $\frac{u_1(x)}{u_2(x)} \neq C$，则 $u_1(x)$, $u_2(x)$ 线性无关.

微分方程的解:
- 不含任意 C 的解 — 特解
- 含相互独立的常数，个数与方程阶数一致的解 — 一通解

二. 一阶微分方程.

种类

解法

(一) 可分离变量型微分方程

1. def - 设 $\frac{dy}{dx} = f(x,y)$. If $f(x,y) = \varphi_1(x) \cdot \varphi_2(y)$
称其为可分离变量.

2. 解法

$$\frac{dy}{dx} = \varphi_1(x) \cdot \varphi_2(y) \Rightarrow \frac{dy}{\varphi_2(y)} = \varphi_1(x)dx$$

$$\Rightarrow \int \frac{dy}{\varphi_2(y)} = \int \varphi_1(x)dx + C$$

例1. 求 $\frac{dy}{dx}=1+x+y^2+xy^2$ 通解.

解. $\frac{dy}{dx}=(1+x)(1+y^2)$

$\Rightarrow \frac{dy}{1+y^2}=(1+x)dx \Rightarrow \int\frac{dy}{1+y^2}=\int(1+x)dx+C \Rightarrow \arctan y=x+\frac{1}{2}x^2+C$

例2. $\frac{dy}{dx}=2xy$ 求通解

解. ① $y=0$ $\frac{dy}{dx}\equiv 0 \Rightarrow y\equiv C$

② $y\neq 0$ $\frac{dy}{y}=2x\,dx$

$\Rightarrow \int\frac{dy}{y}=\int 2x\,dx+C_0 \Rightarrow \ln|y|=x^2+C_0$

$\Rightarrow |y|=e^{C_0}\cdot e^{x^2}$

$\Rightarrow y=e^{C_0}e^{x^2}$ 令 $C=\pm e^{C_0}$ $y=Ce^{x^2}$ $(C\neq 0)$

∴ 通解为 $y=Ce^{x^2}$ (C为任常)

(二) 齐次微分方程

1. def - 设 $\frac{dy}{dx}=f(x,y)$ *

对 $f(x,y)=\varphi(\frac{y}{x})$，称其为齐次.

2. 解法

令 $\frac{y}{x}=u \Rightarrow y=xu \Rightarrow \frac{dy}{dx}=u+x\frac{du}{dx}$ 代入

$\Rightarrow u+x\frac{du}{dx}=\varphi(u) \Rightarrow \frac{du}{\varphi(u)-u}=\frac{dx}{x} \Rightarrow \int\frac{du}{\varphi(u)-u}=\int\frac{dx}{x}+C$

例1. $\frac{dy}{dx}=2\frac{y}{x}-1$，求通解.

解：令 $\frac{y}{x}=u$，则 $u+x\frac{du}{dx}=2u-1$

$\Rightarrow \frac{du}{u-1}=\frac{dx}{x} \Rightarrow \int\frac{du}{u-1}=\int\frac{dx}{x}+\ln C$

$\Rightarrow \ln(u-1)=\ln x+\ln C \Rightarrow u=Cx+1 \Rightarrow y=Cx^2+x$

例2.

$x\,dy-(y+\sqrt{x^2+y^2})\,dx \quad (x>0)$ 求通解.

解：$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}=\frac{y}{x}+\sqrt{1+(\frac{y}{x})^2}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}+\sqrt{1+(\frac{y}{x})^2} \xrightarrow{u=\frac{y}{x}} u+x\frac{du}{dx}=u+\sqrt{1+u^2}$

$\therefore \int\frac{du}{\sqrt{1+u^2}}=\int\frac{dx}{x}+\ln C$

$\Rightarrow \ln(u+\sqrt{1+u^2})=\ln x+\ln C \Rightarrow y+\sqrt{x^2+y^2}=Cx^2$

(三) 一阶齐次线性微方.

1. def. 形如 $\frac{dy}{dx}+p(x)y=0$

2. 解法. $\frac{dy}{dx}=-p(x)\cdot y$

$$\Rightarrow \int\frac{dy}{y}=\int -p(x)dx \Rightarrow \ln y=\ln e^{-\int_0^x p(x)dx}+\ln C$$

$$\boxed{\Rightarrow y=Ce^{-\int p(x)dx} \quad (C\text{为任常})}$$

例1. $\frac{dy}{dx}=2xy$

解: $\frac{dy}{dx}-2xy \Rightarrow y=Ce^{-\int -2xdx}=Ce^{x^2}$ C为任常

例2. $xy'-2y=0$. 求通解

解 $\frac{dy}{dx}-\frac{2y}{x}=0 \Rightarrow y=Ce^{-\int -\frac{2}{x}dx}=Cx^2$

例3. $\frac{dy}{dx}-xy=0$. 求满足 $y|_{x=0}=\pi$ 的特解.

解: $\frac{dy}{dx}-xy=0 \Rightarrow y=Ce^{-\int -xdx}=Ce^{\frac{x^2}{2}}$

而 $y|_{x=0}=\pi \Rightarrow C=\pi$ 即 $y=\pi e^{\frac{x^2}{2}}$

(四) 一阶非齐线性微分方程

1. def 一阶线性 $\frac{dy}{dx} + P(x)y = Q(x)$ (**)

解法：常数变易法 [C → C(x)]

$$\frac{dy}{dx} + P(x)\cdot y = 0 \;:\; y = Ce^{-\int P(x)dx}$$

$$\frac{dy}{dx} + P(x)\cdot y = Q(x) \;:\; y = C(x)e^{-\int P(x)dx}$$

$$C'(x)e^{-\int P(x)dx} - P(x)e^{-\int P(x)dx} + P(x)C(x)e^{-\int P(x)dx} = Q(x)$$

$$\Rightarrow C'(x) = Q(x)e^{\int P(x)dx} \qquad C(x) = \int Q(x)e^{\int P(x)dx}dx + C$$

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$$\Rightarrow y = \left[\int Q(x)e^{\int P(x)dx}dx + C\right]e^{-\int P(x)dx}$$

例1 $\frac{dy}{dx} = 2\frac{y}{x} - 1$

解：$\frac{dy}{dx} - \frac{2}{x}\cdot y = -1$ $\quad P(x) = -\frac{2}{x}$, $Q(x) = -1$

$$\therefore y = \left[\int Q(x)e^{\int P(x)dx}dx + C\right]e^{-\int P(x)dx}$$

$$= \left[\int -e^{\int -\frac{2}{x}dx}dx + C\right]e^{\int \frac{2}{x}dx}$$

$$= [x^{-1} + C]x^2 = x + Cx^2 \quad \text{即 } y = x + Cx^2.$$

例 2. $y'+y\tan x=\cos x$ 通解

$$\frac{dy}{dx}+\tan x\cdot y=\cos x$$

解: $P(x)=\tan x$. $Q(x)=\cos x$

$$\therefore y=\left[\int Q(x)e^{\int P(x)dx}dx+C\right]e^{-\int P(x)dx}$$

$$=\left[\int\cos x\, e^{\int\tan x dx}dx+C\right]e^{-\int\tan x}$$

$$=\left[\int\cos x\, e^{-\ln\cos x}dx+C\right]e^{\ln\cos x}$$

$$=\left[\int 1\,dx+C\right]\cos x=(x+C)\cos x$$

三. 可降阶的高阶微分方程

(一) $y^{(n)}=f(x)$ —— 一直还原

例: $y''=\cos 2x+3x^2$

$$\Rightarrow y'=\tfrac{1}{2}\sin 2x+x^3+C$$

$$\Rightarrow y=-\tfrac{1}{4}\cos 2x+\tfrac{1}{4}x^3+C_1x+C_2$$

(二) $f(x,y',y'')=0$ (缺 y)

令 $y'=P$ $y''=\frac{dP}{dx}$ $f(x,P,\frac{dP}{dx})=0$

↳一阶 $\Rightarrow P=\varphi(x,C_1)=y'\Rightarrow y=\int\varphi(x,C_1)+C_2$

例1. $xy''+2y'=x^2$ 求通解

令 $y'=p$ $\therefore y''=\frac{dp}{dx}$ $\therefore$ $\frac{dp}{dx}+\frac{2}{x}p=x$

$\therefore p=\left(\int xe^{\int\frac{2}{x}dx}dx+C_1\right)e^{-\int\frac{2}{x}dx}$

$=\left(\frac{1}{4}x^4+C\right)x^{-2}=\frac{1}{4}x^2+Cx^{-2}$

即 $y'=\frac{1}{12}x^3+C_1x^{-1}+C_2$

(三) $f(y,y',y'')=0$ (缺 x)

令 $y'=p$. $y''=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}$

$\Rightarrow f\left(y,p,p\frac{dp}{dy}\right)=0$

$\Rightarrow p=\varphi(y,C_1)\Rightarrow\frac{dy}{dx}=\varphi(y,C_1)$

$\Rightarrow\frac{dy}{\varphi(y,C_1)}=dx\Rightarrow\int\frac{dy}{\varphi(y,C_1)}=\int dx+C_2$.

例 2. $yy'' - y'^2 = 0$, $y(0) = y'(0) = 1$. 特解.

解: 设 $y' = p$ $\therefore y'' = p\frac{dp}{dy}$

$\Rightarrow yp\frac{dp}{dy} = p^2$ $\because p \neq 0$ $\therefore y\frac{dp}{dy} - p = 0$

$\Rightarrow \frac{dp}{dy} - \frac{p}{y} = 0$

$\Rightarrow p = C_1 e^{-\int \frac{1}{y} dy} = C_1 y$ $\Rightarrow y' = C_1 y$ 而 $y'(0) = y(0) = 1$

$\therefore C_1 = 1$ $\therefore y' = y$

$\Rightarrow \frac{dy}{dx} - y = 0$ $\Rightarrow y = C_2 e^{-\int dx} = C_2 e^x$

$\because y(0) = 1$ $\therefore C_2 = 1$ $\therefore y = e^x$.

Ⅲ. 高阶线性微分方程.

(一) defs

$$y^{(n)} + a_1(x)y^{(n-1)} + a_2(x)y^{(n-2)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \quad (*)$$

↓ n阶齐次线性微方

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \quad (**)$$

↓ n阶非齐线性微方

若 $f(x) = f_1(x) + f_2(x)$，则

$$\begin{cases} y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f_1(x) & (**)' \\ y^{(n)} + \cdots + a_n(x)y = f_2(x) & (**)'' \end{cases}$$

(二) 解的结构.

1. $y_1(x), \cdots, y_s(x)$ 为 $(*)$ 的解.

$k_1y_1 + k_2y_2 + \cdots + k_sy_s = y$ 为 $(*)$ 的解.

2. $y_1(x) \cdots y_s(x)$ 为 $(**)$ 的解.

① $k_1y_1 + \cdots + k_sy_s$ 为 $(*)$ 的解 $\Leftrightarrow k_1 + k_2 + \cdots + k_s = 0$

② $k_1y_1 + \cdots + k_sy_s$ 为 $(**)$ 的解 $\Leftrightarrow k_1 + k_2 + \cdots + k_s = 1$

3. $y_1(x)$, $y_2(x)$ 分别为 (*), (**) 的解 $\Rightarrow$ y_1+y_2 为 (**) 解.

4. $y_1(x)$, $y_2(x)$ 为 (**) 解 $\Rightarrow$ y_1-y_2 为 (*) 解.

5. $y_1(x)$, $y_2(x)$ 为 (*)', (**)' 的解 $\Rightarrow$ y_1+y_2 为 (*, **) 解.

三. 特殊方程一. $y''+py'+qy=0$ (*) (p, q 为 C)

↳ 二阶常系数齐次线性微方

1° $\lambda^2+p\lambda+q=0$

2° ① $\Delta>0 \Rightarrow \lambda_1\neq\lambda_2 \Rightarrow y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x}$

例1. $y''+y'-2y=0$. 通解

解: $\lambda^2+\lambda-2=0 \Rightarrow \lambda_1=1, \lambda_2=-2$

$\therefore y=C_1e^{x}+C_2e^{-2x}$

② $\Delta=0 \Rightarrow \lambda_1=\lambda_2 \Rightarrow y=C_1e^{\lambda_1 x}+C_2xe^{\lambda_1 x}=(C_1+C_2x)e^{\lambda_1 x}$

例2. $y''-4y'+4y=0$

解. $\lambda^2-4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=2$

$\therefore y=(C_1+C_2x)e^{2x}$

③ $\lambda<0 \Rightarrow \lambda_1, \lambda_2 = \alpha \pm i\beta$

$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$

例3. $y''-2y'+2y=0$

解: $\lambda^2-2\lambda+2=0 \quad \therefore \lambda_{1,2} = \frac{2\pm 2i}{2} = 1\pm i \quad \therefore \alpha=1, \beta=1$

$\therefore y = e^x(C_1\cos x + C_2\sin x)$.

(四) 特殊方程二: $y''+py'+qy=f(x)$ (**) (p, q 常数)

↓

二阶常系数非齐线性微方

(**)的通解 = (*)通解 + (**)特解 ☆

型一: $f(x)=P_n(x)e^{kx}$

例: $y''+y'-2y=(2x+1)e^{2x}$ 求通解.

解: $\lambda^2+\lambda-2=0 \Rightarrow \lambda_1=1, \lambda_2=-2$

1° $y''+y'-2y=0 \Rightarrow y=C_1e^x+C_2e^{-2x}$

1° 令特解 $y_0(x) = (ax+b)e^{2x}$ (按f(x)样子假设, e^{kx}不变, P_n变)

代入 $y_0' = ae^{2x}+(2ax+2b)e^{2x}$

$= (2ax+a+2b)e^{2x} \Rightarrow y_0'' = 2ae^{2x}+(4ax+2a+4b)e^{2x} = (4ax+4a+4b)e^{2x}$

$4ax+4a+4b+2ax+a+2b-2ax-2b=2x+1$

$4ax=2x \Rightarrow a=\frac{1}{2}, \quad 3a+4b=1 \Rightarrow b=-\frac{1}{8}$

$\therefore$ 通解: $y = C_1e^x+C_2e^{-2x}+(\frac{x}{2}-\frac{1}{8})e^{2x}$

② (k 与 λ_1, λ_2 有相同)

例2. $y''-3y'+2y=(2x+1)e^x$ 求通解

解: $\lambda^2-3\lambda+2=0 \Rightarrow \lambda_1=1,\ \lambda_2=2$

1° $y''-3y'+2y=0 \Rightarrow y=C_1e^x+C_2e^{2x}$

2° $y_0(x)=x(ax+b)e^x$ (有相同, 前面再乘 x)

$=(ax^2+bx)e^x$

$y_0'=(ax^2+bx+2ax+b)e^x$

$y_0''=(ax^2+bx+2ax+b+2ax+b+2a)e^x$

$\Rightarrow ax^2+bx+2ax+b+2ax+b+2a-3ax^2-3bx-6ax-3b+2ax^2+2bx=2x+1$

$\therefore b+2a+2a-3b-6a+2b=-2a+2 \quad \therefore a=-1$

$2b+2a-3b=-1 \Rightarrow b=-1$

$\therefore y_0(x)=(-x^2-x)e^x$

$\therefore y=C_1e^x+C_2e^{2x}-(x^2+x)e^x$

③ ($k=\lambda_1=\lambda_2$)

例3. $y''-4y'+4y=(3x+1)e^{2x}$

解: 1° $\lambda^2-4\lambda+4=0 \Rightarrow \lambda_1=\lambda_2=2$

$\therefore y''-4y'+4y=0 \Rightarrow y=(C_1+C_2x)e^{2x}$

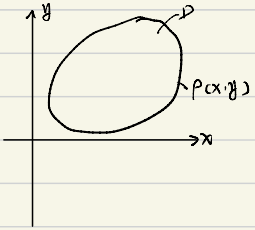
2° 令 $y_0=x^2(ax+b)e^{2x}=(ax^3+bx^2)e^{2x}$

由 $6ax+2b=3x+1 \Rightarrow a=\frac{1}{2},\ b=\frac{1}{2}$

通解 $y=(C_1+C_2x)e^x+(\frac{1}{2}x^3+\frac{1}{2}x^2)e^{2x}$

第八章 二重积分

一. 背景



1. 平面(薄片)质量 m

1° $D \Rightarrow \Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$

2° $\forall (\xi_i, \eta_i) \in \Delta\sigma_i$

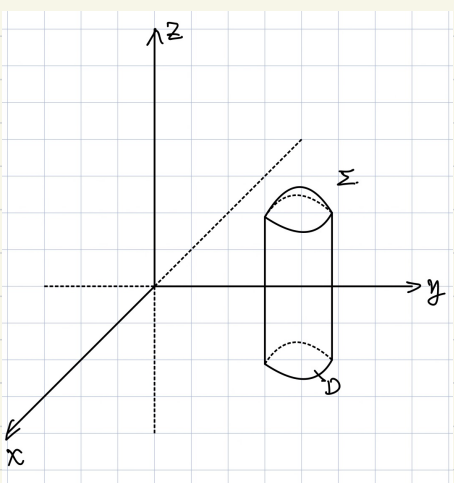
$\Delta m_i \approx \rho(\xi_i, \eta_i)\Delta\sigma_i$

$m \approx \sum_{i=1}^{n} \Delta m_i$

3° 入为 $\Delta\sigma_1, \Delta\sigma_2 \cdots \Delta\sigma_n$ 直径最大者

$m = \lim_{\lambda\to 0} \sum_{i=1}^{n} \rho(\xi_i, \eta_i)\Delta\sigma_i$

2. 空间曲面/柱体



Σ: $z = f(x,y)$ $(x,y) \in D$)

求体积

1° $D \Rightarrow \Delta\sigma_1, \cdots, \Delta\sigma_n$

2° $\forall (\xi_i, \eta_i) \in \Delta\sigma_i$

$\Delta V_i \approx f(\xi_i, \eta_i)\Delta\sigma_i$

$V \approx \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$

3° 入为 $\Delta\sigma_1, \Delta\sigma_2 \cdots, \Delta\sigma_n$ 的直径 max

$V = \lim_{\lambda\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$

二. 定义.

D - xOy面上的有界闭区域，$f(x,y)$在D上有界.

1° $D \Rightarrow \Delta\sigma_1, \Delta\sigma_2, \cdots, \Delta\sigma_n$

2° $\forall (\xi_i, \eta_i) \in \Delta\sigma_i \quad (1 \le i \le n)$

作 $\sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$.

3° λ 为 $\Delta\sigma_1, \Delta\sigma_2 \cdots \Delta\sigma_n$ 直径 max

If $\lim\limits_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$ 存. 称此 I 为 $f(x,y)$ 在D上的二重积分

记为 $\iint\limits_D f(x,y)\,d\sigma$. 即 $\lim\limits_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$.

三. 性质.

① $\iint\limits_D 1\,d\sigma = A$. A为D的面积.

② 设 二重积分积分中值定理.

D为有界闭区域. $f(x,y)$ 在D上连续，则 $\exists (\xi, \eta) \in D$.

使 $\iint\limits_D f(x,y)\,d\sigma = f(\xi, \eta) \cdot A$.

例1. $D: x^2+4y^2 \le t^2 \quad (t>0)$

求 $\lim\limits_{t\to 0} \dfrac{\iint_D e^{x^2}\cos 2y \, d\sigma}{t^2}$.

解: $D = \dfrac{x^2}{t^2} + \dfrac{y^2}{(\frac{t}{2})^2} \le 1$

$\exists (\xi, \eta) \in D$, $\iint_D e^{x^2}\cos 2y \, d\sigma = e^{\xi^2}\cos 2\eta \cdot \pi \cdot t \cdot \frac{t}{2}$

$\therefore$ 原式 $= \frac{\pi}{2} \lim\limits_{t\to 0} e^{\xi^2}\cos 2\eta = \frac{\pi}{2} \times 1 \times 1 = \frac{\pi}{2}$.

③ ☆

(1). D 关于左右对称 (关于 y 轴对称, 或关于 x 对轴). 记 D_1

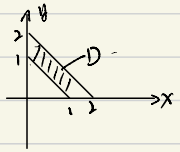
If $f(-x,y) = -f(x,y)$, 则 $\iint_D f(x,y)\, d\sigma = 0$

If $f(-x,y) = f(x,y)$, 则 $\iint_D f(x,y)\, d\sigma = 2\iint_{D_1} f(x,y)\, d\sigma$.

(2) 设 D 关于 $y=x$ 对称, 则

$\iint_D f(x,y)\, d\sigma = \iint_D f(y,x)\, d\sigma$.

例2. 求 $I = \iint_D \frac{x^3}{x^2+y^2} \, d\sigma$.



$I = \iint_D \frac{y^3}{x^2+y^2}$

$\therefore 2I = \iint_D d\sigma = \frac{3}{2} \Rightarrow I = \frac{3}{4}$

四. 积分方法

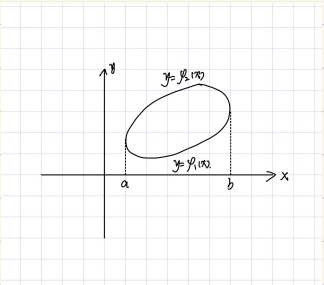
(一) 直角坐标法. 对 $\iint_D f(x,y)\,d\sigma$.

1. $D=\{(x,y)\mid a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$

↓

x型区域

$$\iint_D f(x,y)\,d\sigma=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

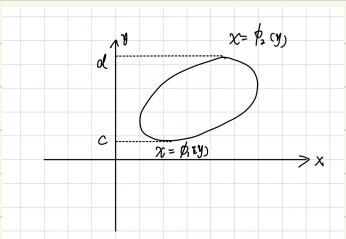


2. $D=\{(x,y)\mid c\le y\le d,\ \phi_1(y)\le x\le \phi_2(y)\}$

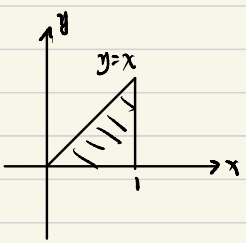
↓

Y型区域

$$\iint_D f(x,y)\,d\sigma=\int_c^d dy\int_{\phi_1(y)}^{\phi_2(y)} f(x,y)\,dx.$$



例1. $I=\iint_D x^2y\,d\sigma$



解. 法1.

$D=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le x\}$

$$I=\int_0^1 x^2dx\int_0^x y\,dy=\frac{1}{2}\int_0^1 x^4dx=\frac{1}{10}$$

法2.

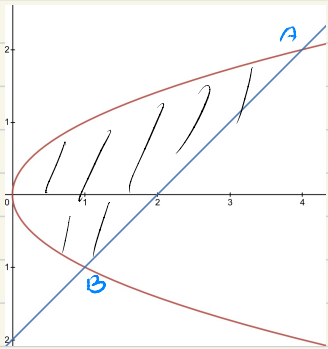
$D=\{(x,y)\mid 0\le y\le 1,\ y\le x\le 1\}$

$$I=\int_0^1 y\,dy\int_y^1 x^2dx=\frac{1}{3}\int_0^1 y(1-y^3)\,dy=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{1}{10}.$$

例 2.

例2 计算 $\iint_D (x+y)\,dx\,dy$，其中 D 由 $x=y^2$ 与 $y=x-2$ 围成.

解：$\begin{cases} x=y^2 \\ x=y+2 \end{cases} \Rightarrow A(4,2),\ B(1,-1)$



$$\iint_D (x+y)\,dx\,dy = \int_{-1}^{2} dy \int_{y^2}^{y+2} (x+y)\,dx$$

$$= \int_{-1}^{2} dy \cdot \left(\frac{1}{2}x^2+xy\right)\Big|_{y^2}^{y+2}$$

$$= 9\frac{9}{20}.$$

例 3. 计算 $\int_0^1 dy \int_y^1 e^{x^2}dx$ (大坑)

Notes.

① $x^{2n}e^{\pm x^2}dx$

② $e^{\frac{k}{x}}dx$

③ $\cos\frac{k}{x}dx$, $\sin\frac{k}{x}dx$

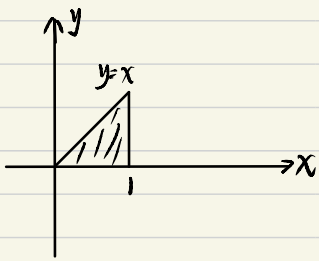
} 算不了. 交换次序.

解.

$$I = \int_0^1 dx \int_0^x e^{x^2}dy$$

$$= \int_0^1 e^{x^2}dx \int_0^x 1\,dy \Rightarrow \int_0^1 xe^{x^2}dx$$

$$= \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{e-1}{2}.$$



(二) 极坐标法

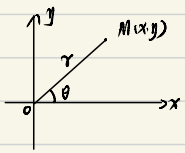
1°. 特征

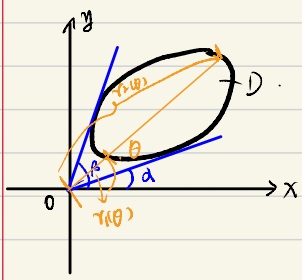
① D 边界曲线含 x^2+y^2,

② $f(x,y)$ 含 x^2+y^2

2° 变换.

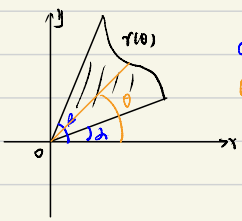
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$





$\alpha \le \theta \le \beta$

$r_1(\theta) \le r \le r_2(\theta)$



$\alpha \le \theta \le \beta$

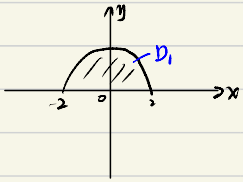
$0 \le r \le r(\theta)$

3°. $d\sigma = dx\,dy \Rightarrow$ ① 取 $[\theta, \theta+d\theta] \subset [\alpha, \beta] \Rightarrow [r, r+dr]$

$\Rightarrow d\sigma = r\,dr\,d\theta$

$$\iint_D f(x,y)\,d\sigma = \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} r \cdot f(r\cos\theta, r\sin\theta)\,dr$$

例1. $I=\iint_D x^2\,d\sigma$



解: $I=2\iint_{D_1} x^2\,d\sigma$

法一. 令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $(0\le\theta\le\frac{\pi}{2},\ 0\le r\le 2)$

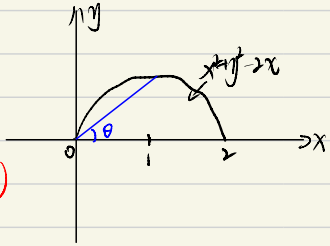
$I=2\int_0^{\frac{\pi}{2}}\cos^2\theta\int_0^2 r^3\,dr = 8\times\frac{1}{2}\times\frac{\pi}{2}=2\pi.$

法二.

$I=2\iint_{D_1} y^2\,d\sigma$

$I=\iint_{D_1}(x^2+y^2)\,d\sigma = \int_0^{\frac{\pi}{2}}d\theta\int_0^2 r^3\,dr = 4\cdot\frac{\pi}{2}=2\pi.$

例2. $I=\iint_D x^2\,d\sigma$

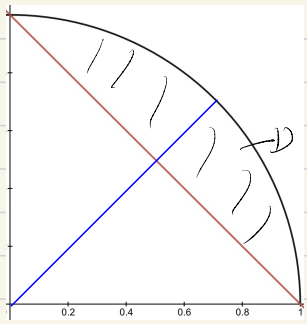


令 $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $(0\le\theta\le\frac{\pi}{2},\ 0\le r\le 2\cos\theta)$

$I=\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta\int_0^{2\cos\theta} r^3\,dr = 4\int_0^{\frac{\pi}{2}}\cos^6\theta\,d\theta = 4\times\frac{5}{6}\times\frac{3}{4}\times\frac{2}{2}\times\frac{1}{2}$

$=\frac{5}{8}\pi$

例 3. $I = \iint_D \frac{1}{\sqrt{x+y}}\, d\sigma$



令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$\left(0 \le \theta \le \frac{\pi}{2},\ \frac{1}{\sin\theta+\cos\theta} \le r \le 1\right)$

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} 1\, dr$$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{\sin\theta+\cos\theta}\right) d\theta$$

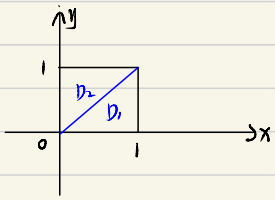
$$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta+\cos\theta}\, d\theta = \frac{\pi}{2} - \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \sec\left(\theta - \frac{\pi}{4}\right) d\left(\theta - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{1}{\sqrt{2}} \ln\left|\sec\theta + \tan\theta\right| \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - \sqrt{2} \ln\left|\sec\theta + \tan\theta\right| \Big|_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{2} - \sqrt{2} \ln\left(\sqrt{2}+1\right)$$

例 4. $I=\iint_D \sqrt{|x-y|}\,d\sigma$



解:

$$I=\iint_{D_1}\sqrt{x-y}\,dxdy+\iint_{D_2}\sqrt{y-x}\,dxdy$$

$$\iint_{D_1}\sqrt{x-y}\,d\sigma=\int_0^1 dx\int_D^x (x-y)^{\frac{1}{2}}\,d(x-y)$$

$$=-\frac{2}{3}\int_0^1 (x-y)^{\frac{3}{2}}\Big|_0^x=-\frac{2}{3}\int_0^1 (0-x^{\frac{3}{2}})=\frac{2}{3}\cdot\frac{2}{5}\cdot x^{\frac{5}{2}}\Big|_0^1=\frac{4}{15}$$

$$\iint_{D_2}\sqrt{y-x}\,d\sigma=-\int_0^1 dy\int_0^y (y-x)^{\frac{1}{2}}\,d(y-x)=\frac{4}{15}$$

$$\text{所以 } I=\frac{8}{15}.$$

第九章 无穷级数

Part 1. 常数项级数

一. defs.

1. $\{a_n\}$ 为常数项列，$\sum_{n=1}^{\infty} a_n$ 称为常数项级数.

2. $S_n = a_1+a_2+\cdots+a_n$ — 部分和

Notes:

① S_n 与 $\sum_{n=1}^{\infty} a_n$ 不同. ② $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$

If $\lim_{n\to\infty} S_n = S$，则 $\sum_{n=1}^{\infty} a_n = S$；

If $\lim_{n\to\infty} S_n$ 不∃，则发散.

二. 性质

① $\sum_{n=1}^{\infty} a_n = A$，$\sum_{n=1}^{\infty} b_n = B$，则 $\sum_{n=1}^{\infty} a_n \pm b_n = A \pm B$

② $\sum_{n=1}^{\infty} a_n = S$，则 $\sum_{n=1}^{\infty} k a_n = kS$ (If $k\neq 0$, 则 $\sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} k a_n$ 敛散性相同)

③ 级数 +, −, 改 有限项，敛散性不变.

加了括号的级数与原级数不一定相同

④ 加括号提升收敛性

① If $\sum_{n=1}^{\infty} a_n$ 收敛，$\Rightarrow$ 添加括号后收敛

② 加括号发散 $\Rightarrow$ 原式发散.

例. 设 $\sum_{n=1}^{\infty} a_n$ 收敛, 问 $\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n})$?

解 $\sum_{n=1}^{\infty} a_n$ 本身收敛, 则

$$\sum_{n=1}^{\infty}(a_{2n-1}+a_{2n}) = (a_1+a_2)+(a_3+a_4)+\cdots \text{ 收敛}$$

① (必要条件) $\sum_{n=1}^{\infty} a_n$ 收敛 $\not\Rightarrow$ $\lim_{n\to\infty} a_n = 0$

证 "$\Rightarrow$" $S_n = a_1+a_2+\cdots+a_n$

$\because \sum_{n=1}^{\infty} a_n$ 收敛 $\therefore \lim_{n\to\infty} S_n$ 存在. 设 $\lim_{n\to\infty} S_n = S$

$\therefore a_n = S_n - S_{n-1}$ $\therefore \lim_{n\to\infty} a_n = S - S = 0$

证 "$\not\Leftarrow$" $\sum_{n=1}^{\infty} \frac{1}{n}$, $\lim_{n\to\infty} a_n = 0$

↳ 发散

Notes:

若 $\sum a_n$ 收敛 $\Rightarrow$ $\begin{cases} \lim S_n \exists \\ \lim a_n = 0 \end{cases}$

三. 两个重要级数

1. p-级数 - $\sum_{n=1}^{\infty}\frac{1}{n^p}$

$\begin{cases} \text{收敛} & p>1 \\ \text{发散} & p\le 1 \end{cases}$

当 $n=1$ 时，即 $\sum_{n=1}^{\infty}\frac{1}{n}$ - 调和级数，发散

2. 几何级数 - $\sum_{n=1}^{\infty}aq^n \quad (a\neq 0)$

① If $|q|\ge 1$，则 $\sum_{n=1}^{\infty}aq^n$ 发散

② If $|q|<1$，则 $\sum_{n=1}^{\infty}aq^n = a\frac{\text{第一项}}{1-q}$

四. 第一类 - 正项级数

(一) def - $\sum_{n=1}^{\infty}a_n \ (a_n\ge 0)$，称为正项级数

Note: 设 $\sum_{n=1}^{\infty}a_n$ 为正项级数

① $S_1\le S_2\le S_3\cdots$ 即 S_n 单增

② $\begin{cases} S_n \text{无上界} \Rightarrow \lim\limits_{n\to\infty}S_n=+\infty, \text{发散} \\ S_n \text{有上界} \Rightarrow \lim\limits_{n\to\infty}S_n=M, \text{收敛} \end{cases}$

(二) 判别法

法一：比较法

Th 1. $a_n \geq 0,\ b_n \geq 0$

① $a_n \leq b_n$ 且 $\sum_{n=1}^{\infty} b_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 收敛

② $a_n \geq b_n$ 且 $\sum_{n=1}^{\infty} b_n$ 发散 $\Rightarrow \sum_{n=1}^{\infty} a_n$ 发散

例 1. 判断 $\sum_{n=1}^{\infty} \sin\frac{\pi}{2^n}$ 的敛散性.

解. $\because x \geq 0$ 时, $\sin x < x$ $\therefore 0 \leq \sin\frac{\pi}{2^n} < \frac{\pi}{2^n}$

$\because \sum_{n=1}^{\infty} \frac{\pi}{2^n}$ 收敛 $\therefore \sum_{n=1}^{\infty} \sin\frac{\pi}{2^n}$ 收敛.

例 2. 判断 $\sum_{n=1}^{\infty} \frac{1}{\int_0^n \sqrt[4]{1+x^4}\,dx}$ 敛散性

解.

$$\int_0^n \sqrt[4]{1+x^4} \geq \int_0^n x\,dx = \frac{n^2}{2}$$

$$\therefore \frac{1}{\int_0^n \sqrt[4]{1+x^4}\,dx} \leq \frac{2}{n^2} \quad \text{而} \sum_{n=1}^{\infty} \frac{2}{n^2} \text{ 收敛}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{\int_0^n \sqrt[4]{1+x^4}\,dx} \text{ 收敛.}$$

例 3. $a_n \le b_n \le c_n$，$\sum_{n=1}^{\infty} a_n$，$\sum_{n=1}^{\infty} c_n$ 收敛

证 $\sum_{n=1}^{\infty} b_n$ 收敛

证：$a_n \le b_n \le c_n \Rightarrow 0 \le b_n - a_n \le c_n - a_n$

$\sum_{n=1}^{\infty} a_n$，$\sum_{n=1}^{\infty} c_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} (c_n - a_n)$ 收敛

$\Rightarrow \sum_{n=1}^{\infty} b_n - a_n$ 收敛

$\because \sum_{n=1}^{\infty} a_n$ 收敛 $\therefore \sum_{n=1}^{\infty} b_n$ 收敛

Th 1' $a_n > 0$，$b_n > 0$

If $\lim_{n\to\infty} \frac{b_n}{a_n} = l$ $(0 < l < +\infty)$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ 与 $\sum_{n=1}^{\infty} b_n$ 敛散性相同.

例 4 $\sum_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$ 敛散性？

解：$\because x > 0$ 时 $\ln(1+x) < x$ $\therefore \ln(1+\frac{1}{n}) < \frac{1}{n}$

$\Rightarrow \frac{1}{n} - \ln(1+\frac{1}{n}) > 0$

$\because \lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \frac{1}{2}$ $\therefore \lim_{n\to\infty} \frac{\frac{1}{n} - \ln(1+\frac{1}{n})}{\frac{1}{n^2}} = \frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n} - \ln(1+\frac{1}{n})$ 与 $\sum_{n=1}^{\infty} \frac{1}{n^2}$ 敛散性相同

故 $\sum_{n=1}^{\infty} [\frac{1}{n} - \ln(1+\frac{1}{n})]$ 收敛

方法二：比值法

Th 2. $a_n>0$，$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\rho$ $\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ ? & \rho=1\end{cases}$

Notes. 若 a_n 中含 $n!$，用比值法

例 5. $\sum\limits_{n=1}^{\infty}\frac{2^n n!}{n^n}$?

解. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{2^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n\cdot n!}$

$=2\lim\limits_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}=\frac{2}{e}=\rho<1\Rightarrow$ 收敛.

方法三：根值法

Th 3. $a_n>0$，$\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\rho$ $\begin{cases}\text{收敛} & \rho<1\\ \text{发散} & \rho>1\\ ? & \rho=1\end{cases}$

例 6. $\sum\limits_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$?

解：$\lim\limits_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}=\rho<1\Rightarrow$ 收敛.

五. 第二类 — 交错级数

(一) def — $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ $(a_n > 0,\ n = 1, 2, \cdots)$

or $\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - \cdots$

(二) 判别法

Th 对 $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ $(a_n > 0,\ n = 1, 2, \cdots)$

If $\begin{cases} ① \{a_n\} \text{单减} \\ ② \lim\limits_{n\to\infty} a_n = 0 \end{cases} \Rightarrow$ 收敛 且 $S \le a_1$

如 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$

① $\frac{1}{\sqrt{n}} \downarrow$, 且 $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ $\Rightarrow$ 收敛.

Q1. $\sum_{n=1}^{\infty} a_n$ 收敛 $\overset{?}{\Longrightarrow}$ $\sum_{n=1}^{\infty} a_n^2$ 收敛 (X)

$a_n = \frac{(-1)^n}{\sqrt{n}}$ 收敛. $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ 发散.

Q2. $\sum_{n=1}^{\infty} a_n$ ($a_n > 0$) 收敛 $\overset{?}{\Longrightarrow}$ $\sum_{n=1}^{\infty} a_n^2$ 收敛 (√)

证: $\sum_{n=1}^{\infty} a_n$ 收敛, $\lim_{n\to\infty} a_n = 0$

取 $\varepsilon = 1 > 0$, $\exists N > 0$, 当 $n > N$, $|a_n - 0| < 1$ $\Rightarrow a_n < 1$

$0 \le a_n^2 \le a_n < 1$ $\because \sum_{n=1}^{\infty} a_n$ 收敛 $\Rightarrow \sum_{n=1}^{\infty} a_n^2$ 收敛.

例7 $\sum_{n=1}^{\infty} \sin \sqrt{n^2+1}\,\pi$?

解: $a_n = \sin \sqrt{n^2+1}\,\pi = \sin[n\pi + (\sqrt{n^2+1} - n)\pi]$

$= (-1)^n \sin\left[\frac{\pi}{\sqrt{n^2+1}+n}\right]$

$\because \sin \frac{\pi}{\sqrt{n^2+1}+n} > 0$ $\therefore$ 原级数为交错级数.

$\because \left\{\sin \frac{\pi}{\sqrt{n^2+1}+n}\right\} \downarrow$ 且 $\lim_{n\to\infty} a_n = 0$ $\Rightarrow$ 收敛.

六. 任意级数 - 条件收敛与绝对收敛

(一) def.

① 若 $\sum_{n=1}^{\infty} c_n$ 收敛, 但 $\sum_{n=1}^{\infty} |c_n|$ 发散, 称 $\sum_{n=1}^{\infty} c_n$ 条件收敛.

② 若 $\sum_{n=1}^{\infty} |c_n|$ 收敛, 则 $\sum_{n=1}^{\infty} c_n$ 绝对收敛.

(二) 关系

Th $\sum_{n=1}^{\infty} c_n$ 绝对收敛 $\Rightarrow$ $\sum_{n=1}^{\infty} c_n$ 收敛.

例 8. $\sum_{n=1}^{\infty} \sin\sqrt{n^2+1}\,\pi$, 条件还是绝对.

解: $c_n = \sin\sqrt{n^2+1}\,\pi = (-1)^n \sin\frac{\pi}{\sqrt{n^2+1}+n}$

$\because |c_n| \sim \sin\frac{\pi}{\sqrt{n^2+1}+n} \sim \sin\frac{\pi}{2n}$ 而 $\sum_{n=1}^{\infty} \sin\frac{\pi}{2n}$ 发散

$\therefore$ 条件收敛.

Part II. 幂级数.

(一) def

1. $\{u_n(x)\}$ - 函数列. $\sum_{n=1}^{\infty}\{u_n(x)\}$ 为函数项级数.
   当 $x=x_0$ 时. $\sum_{n=1}^{\infty}u_n(x_0)$ 收敛. 称 x_0 为收敛点.
   当 $x=x_1$ 时 $\sum_{n=1}^{\infty}u_n(x_1)$ 发散. 称 x_1 为发散点.
   一切收敛点的集合称为收敛域.

2. $\sum_{n=0}^{\infty}a_nx^n = a_0+a_1x+a_2x^2+\cdots$

   $\sum_{n=0}^{\infty}a_n(x-x_0)^n = a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots$

   } $\Rightarrow$ 幂级数.

3. $\sum_{n=0}^{\infty}a_nx^n$

   $\exists R\geq 0$. $|x|<R$: $\sum_{n=0}^{\infty}a_nx^n$ 绝对收敛

   $|x|>R$: 发散

   $|x|=R$: ?

   则 R - 收敛半径. , $(-R,R)$ 收敛区间.

二. 收敛半径. 收敛域.

Th. 对 $\sum_{n=0}^{\infty}a_nx^n$. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho$. 或 $\lim_{n\to\infty}\sqrt[n]{|a_n|}=\rho$

① $\rho=0$, $\Rightarrow R=+\infty$

② $\rho=+\infty$ $\Rightarrow R=0$

③ $0<\rho<+\infty$ $\Rightarrow R=\frac{1}{\rho}$.

例1. $\sum_{n=1}^{\infty} n! x^n$?

解: $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = +\infty \Rightarrow R=0 \Rightarrow$ 收敛域 $\{0\}$.

例2. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$?

解. $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left(\frac{1}{n+1}\right) = 0 \Rightarrow R=+\infty \Rightarrow$ 收敛域 $(-\infty,+\infty)$

例3. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n\cdot 2^n}$?

解: $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \frac{1}{2} \Rightarrow R=2$.

$x-1=-2$ 时. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n\cdot 2^n} = \sum_{n=1}^{\infty} (-1)^n/n$ 收敛.

$x-1=2$ 时. $\sum_{n=1}^{\infty} \frac{2^n}{n\cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ 发散

$\therefore -2 \le x-1 < 2 \Rightarrow -1 \le x < 3$ 即收敛域 $[-1,3)$.

三. 分析性质

对 $\sum_{n=0}^{\infty} a_n x^n \quad x \in (-R, R)$

$\parallel$

$S(x)$ — 和函数.

Th 1. (逐项可导性) 对 $\sum_{n=0}^{\infty} a_n x^n$. $x \in (-R, R)$, 则 $S'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)'$

即. $S'(x) = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ — 收敛半径=R

Th 2. (逐项可积性) 对 $\sum_{n=1}^{\infty} a_n x^n$, $x \in (-R, R)$ 有

$$\int_0^x S(x)\,dx = \int_0^x \left(\sum_{n=1}^{\infty} a_n x^n\right) dx = \sum_{n=1}^{\infty} \int_0^x a_n x^n\,dx$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{— 收敛半径 } R.$$

四. 任务一: $f(x) \Rightarrow \sum_{n=0}^{\infty} a_n (x-x_0)^n$

(一) 直接法 (公式法)

Taylor 公式: 若 $f(x)$ 在 $x=x_0$ 邻域内有 $(n+1)$ 阶导数, 则

$$f(x) = p_n(x) + R_n(x), \quad p_n(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}}{n!}(x-x_0)^n$$

$$R_n(x) = \begin{cases} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \\ o((x-x_0)^n) \end{cases}$$

If $x_0=0$ $f(x)=f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+R_n(x)$

① e^x ② $\sin x$ ③ $\cos x$ ④ $\frac{1}{1-x}$ ⑤ $\frac{1}{1+x}$ ⑥ $\ln(1+x)$

Th. 若 $f(x)$ 在 $x=x_0$ 邻域内任意阶可导. 则

$f(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^n$ $c_n=\frac{f^{(n)}(x_0)}{n!}$ $(n=0,1,2\cdots)$.

If $x_0=0$ $\therefore f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

记:

① $e^x=1+x+\frac{x^2}{2!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ $(-\infty<x<+\infty)$

② $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$ $(-\infty<x<+\infty)$

③ $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2n!}$ $(-\infty<x<+\infty)$

④ $\frac{1}{1-x}=1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n$ $(-1<x<1)$

⑤ $\frac{1}{1+x}=1-x+x^2-\cdots=\sum_{n=0}^{\infty}(-1)^nx^n$ $(-1<x<1)$

⑥ $\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$ $(-1<x\le 1)$

$\Rightarrow$ $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots=\ln 2$.

$$\ln(1+x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad (-1 < x \le 1)$$

$$\Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} \cdots = \sum_{n=1}^{\infty} \frac{1}{n} x^n \quad (-1 \le x < 1)$$

(二)间接法. $\begin{cases} \text{分析性质} \\ ① \sim ⑦. \end{cases}$

例1. $f(x) = \frac{1}{x^2-1}$ 展成 $(x-2)$ 的幂级数.

解: $f(x) = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$

$$\frac{1}{x-1} = \frac{1}{1+(x-2)} = \sum_{n=0}^{\infty} (-1)^n (x-2)^n \quad (1 < x < 3)$$

$$\frac{1}{x+1} = \frac{1}{(x-2)+3} = \frac{1}{3} \cdot \frac{1}{1+\frac{x-2}{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-2)^n \quad (-1 < x < 5)$$

$$\therefore f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(1 + \frac{1}{3^{n+1}}\right)(x-2)^n \quad (1 < x < 3)$$

例2. $f(x) = \frac{5x-1}{x^2-x-2}$ 展成 $(x-1)$ 的幂级数.

解: $f(x) = \frac{5x-1}{(x+1)(x-2)} = \frac{A}{x+1} - \frac{B}{x-2}$

$$\begin{cases} A+B=5 \\ -2A+B=-1 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=3 \end{cases}$$

$$f(x) = \frac{5x-1}{x^2-x-2} = 2\frac{1}{x+1} + 3 \cdot \frac{1}{x-2}$$

$$\frac{1}{x+1} = \frac{1}{2} \cdot \frac{1}{1+\frac{x-1}{2}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-1)^n \quad (-1 < x < 3)$$

$$\frac{1}{x-2} = \frac{1}{-1+(x-1)} = (-1)\frac{1}{1-(x-1)} = -\sum_{n=1}^{\infty} (x-1)^n \quad (0 < x < 2)$$

$f(x) = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{2^n} - 3\right](x-1)^n \qquad (0<x<2)$

3. $f(x) = \arctan x$ 展成 x 幂级数.

解: $f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad (-1<x<1)$

$\because f(0)=0 \quad \therefore f(x) = f(x) - f(0) = \int_0^x f'(x)\,dx = \int_0^x \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad (-1 \le x \le 1)$

$\therefore \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad x \in [-1,1]$

二) 求和函数. $\begin{cases} \text{分析性质} \\ ① \sim ⑦ \end{cases}$

Case 1. $\sum P(n) x^n$ $\begin{cases} ④ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (-1<x<1) \\ ⑤ \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x} \quad (-1<x<1) \end{cases}$

例 1. $\sum_{n=0}^{\infty} (n+2) x^n$, 求 $S(x)$

解: 1° $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+3}{n+2} = 1 \Rightarrow R=1$

2° $x=\pm 1$ 时, $(n+2)(\pm 1)^n \not\to 0 \ (n\to\infty)$ $\therefore$ 收敛域 $(-1,1)$

3° $S(x) = \sum_{n=1}^{\infty} nx^n + 2\sum_{n=0}^{\infty} x^n$

$= x\sum_{n=1}^{\infty} nx^{n-1} + 2\sum_{n=0}^{\infty} x^n$

$= x\sum_{n=1}^{\infty} (x^n)' + \frac{2}{1-x}$

$= x\left(\sum_{n=1}^{\infty} x^n\right)' + \frac{2}{1-x} = x\left(\frac{x}{1-x}\right)' + \frac{2}{1-x} = \frac{x}{(1-x)^2} + \frac{2}{1-x} = \frac{2-x}{(1-x)^2}$

x 看成常数.

2. $\sum_{n=0}^{\infty} n^2 x^n$ 求 S_x

1° $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = 1$ ∴ $R=1$

$x=\pm 1$ 时 $n^2(\pm 1)^n \not\to 0$ $(n\to\infty)$ ∴ 收敛域 $(-1,1)$

2° $S(x) = \sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} [n(n-1)+n] x^n$

$= x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} nx^{n-1}$

$= x^2\left(\sum_{n=2}^{\infty} x^n\right)'' + x\left(\sum_{n=1}^{\infty} x^n\right)'$

$= x^2\left(\frac{x^2}{1-x}\right)'' + x\left(\frac{x}{1-x}\right)'$

Case 2. $\sum \frac{x^n}{p(n)}$ $\begin{cases} ⑥,⑦ \\ \text{去分母} \end{cases}$

例3. $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$ 求 $S(x)$

解. 1° $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow R=1$

$x=\pm1$ 时, $\left|\frac{(\pm1)^n}{n(n+1)}\right| \sim \frac{1}{n^2}$ 且 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 收敛.

$\therefore \sum_{n=1}^{\infty}\left|\frac{(\pm1)^n}{n(n+1)}\right|$ 收敛 $\therefore x\in[-1,1]$

2° $S(x) = \sum_{n=1}^{\infty}\frac{x^n}{n(n+1)} = \sum_{n=1}^{\infty}\frac{1}{n}x^n - \sum_{n=1}^{\infty}\frac{1}{n+1}x^n$

① $S(0)=0$

↳ps: 分子次数很不上分母, 必讨论 $S(0)$

② $x\neq0$ 时. $S(x) = -\ln(1-x) - \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}$

$= -\ln(1-x) - \frac{1}{x}\sum_{n=2}^{\infty}\frac{x^n}{n} = -\ln(1-x) - \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n} - x\right)$

$= \left(\frac{1}{x}-1\right)\ln(1-x)+1$ $(-1\le x<1$, 且 $x\neq0)$

③ $x=1$ 时. $S(1) = \lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$

$$S(x) = \begin{cases} 0 & x=0 \\ 1 & x=1 \\ \left(\frac{1}{x}-1\right)\ln(1-x)+1 & -1\le x<1 \text{ 且 } x\neq0 \end{cases}$$

4. $\sum_{n=1}^{\infty} \frac{x^{2n}}{n(2n-1)}$，求 $S(x)$

解：

1° $R=1$

$x=\pm 1$ 时，$\left|\frac{(\pm 1)^{2n}}{n(2n-1)}\right| \sim \frac{1}{2n^2}$ 且 $\sum_{n=1}^{\infty}\frac{1}{2n^2}$ 收敛 $\therefore \sum_{n=1}^{\infty}\left|\frac{(\pm 1)^{2n}}{n(2n-1)}\right|$ 收敛

$\therefore x\in[-1,1]$

2° $S(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{n(2n-1)} = 2\left(\sum_{n=1}^{\infty}\frac{x^{2n}}{2n-1}-\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}\right)$

$=2[S_1(x)-S_2(x)]$

$S_1(x) = x\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} = x\sum_{n=1}^{\infty}\int_0^x x^{2n-2}dx$

$= x\int_0^x\left(\sum_{n=1}^{\infty}x^{2n-2}\right)dx = x\int_0^x\frac{1}{1-x^2}dx = -\frac{x}{2}\int_0^x\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$

$=-\frac{x}{2}\ln\left|\frac{x-1}{x+1}\right| = -\frac{x}{2}\ln\frac{1-x}{1+x}$ $(-1<x<1)$

$S_2(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(x^2)^n}{n} = -\frac{1}{2}\ln(1-x^2)$ $(-1<x<1)$

$\therefore S(x) = \ln(1-x^2)-x\ln\frac{1-x}{1+x}$ $(-1<x<1)$

$S(\pm 1) = \sum_{n=1}^{\infty}\frac{1}{n(2n-1)} = 2\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right) = 2\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\right) = 2\ln 2$

$\therefore S(x)=\begin{cases} 2\ln 2 & x=1,-1 \\ \ln(1-x^2)-x\ln\frac{1-x}{1+x} & x\in(-1,1) \end{cases}$

5. $\sum_{n=0}^{\infty}(-1)^n \frac{4n^2+4n+2}{2n+1}x^{2n}$ 求 $S(x)$

解.

1° $R=1$ $x=\pm1$ 时 $(-1)^n \frac{2n(n+1)^2+1}{2n+1} \not\to 0$ $(n\to\infty)$ $\therefore x\in(-1,1)$

2° $S(x)=\sum_{n=0}^{\infty}(-1)^n\left(2n+1+\frac{1}{2n+1}\right)x^{2n}$

$=\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n}+\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n}$

① $S(0)=2$;

② $x\neq0$ 时, $S(x)=\left[\sum_{n=0}^{\infty}(-1)^n x^{2n+1}\right]'+\frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$

$=\left(\frac{x}{1+x^2}\right)'+\frac{1}{x}\int_0^x\left[\sum_{n=0}^{\infty}(-1)^n x^{2n}\right]dx$

$=\left(\frac{x}{1+x^2}\right)'+\frac{1}{x}\int_0^x\frac{1}{1+x^2}dx$

$=\frac{1-x^2}{(1+x^2)^2}+\frac{1}{x}\arctan x$

$$S(x)=\begin{cases}2 & x=0\\ \frac{1-x^2}{(1+x^2)^2}+\frac{1}{x}\arctan x & 0<|x|<1 \text{ 且 } x\neq0\end{cases}$$

Part III 傅里叶级数

单一周期信号：$a_n \cos n\omega t + b_n \sin n\omega t$

Q1：$f(x)$ 可否分解为 $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$？

$a_0 = ?$ $a_n = ?$ $b_n = ?$

$\frac{a_0}{2}$ - 直流成分

$a_1 \cos x + b_1 \sin x$ - 一次谐波

$a_2 \cos 2x + b_2 \sin 2x$ - 二次谐波

Q2：$f(x)$ 与 $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ 何关系？

一、三角函数系及正交性

1. $\cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots$

↓

三角函数系

1° $\int_{-\pi}^{\pi} 1 \cdot \cos nx \, dx = 0$ $(n = 1, 2, 3, \cdots)$

2° $\int_{-\pi}^{\pi} 1 \cdot \sin nx \, dx = 0$ $(n = 1, 2, 3, \cdots)$

3° $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ $(m \geq 1, n \geq 1)$

↳ $\frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$

4° $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$ $(m \geq 1, n \geq 1$ 且 $m \neq n)$

5° $\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \, dx = 0$ $(m \geq 1,\ n \geq 1,\ m \neq n)$

6° $\int_{-\pi}^{\pi} 1^2 dx = 2\pi$

7° $\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$ $(n = 1, 2, \cdots)$

8° $\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$ $(n = 1, 2, \cdots)$

(二) 周期为 2π 的函数展成傅里叶级数.

设 $f(x)$ 以 2π 为周期

Q1: $f(x)$ 可否分解为 $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$?

$a_0 = ?$ $a_n = ?$ $b_n = ?$

Q2: $f(x)$ 与上述展开的关系

Th. (Dirichlet 充分条件)

设 $f(x)$, $T = 2\pi$. If
① $f(x)$ 在 $[-\pi, \pi)$ 连续 / 有有限第一类间断点
② $f(x)$ 在 $[-\pi, \pi]$ 内仅有有限个极值点

则
1° $f(x)$ 可展成 $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ 且 $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$
$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx$ $(n = 1, 2, \cdots)$
$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx$ $(n = 1, 2, \cdots)$

2° i) x为$f(x)$连续点

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=f(x);$$

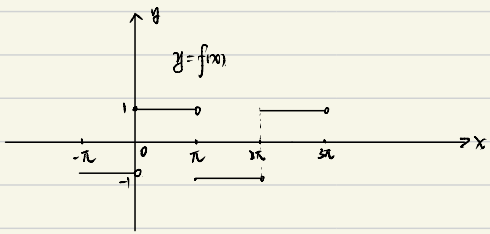
ii) x为$f(x)$间断点.

则 $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{f(x-0)+f(x+0)}{2}$

例1. $f(x)$以2π为周期, $f(x)$在$[-\pi,\pi)$上表达式为

$$f(x)=\begin{cases}-1 & -\pi\le x<0\\ 1 & 0\le x<\pi\end{cases}$$

将$f(x)$展开成傅里叶级数.



解: 1° 作$y=f(x)$图

$x=k\pi$ $(k\in\mathbb{Z})$为$f(x)$间断点,

2° $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=0$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=0$ $(n=1,2,\cdots)$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}\sin nx\,dx=\frac{2}{n\pi}\cos nx\Big|_0^{\pi}=-\frac{2}{n\pi}[(-1)^n-1]=\frac{2}{n\pi}[1-(-1)^n]$

$$=\begin{cases}\frac{4}{n\pi} & n=2k-1\\ 0 & n=2k\end{cases}\quad(k\in\mathbb{Z}^+)$$

3° $f(x)=\frac{4}{\pi}\left(\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\cdots\right)$

$=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$ $(-\infty<x<+\infty$ 且 $x\neq k\pi$, $k\in\mathbb{Z})$

4° $x=k\pi$ 时

$$\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}=\frac{f(k\pi-0)+f(k\pi+0)}{2}=0$$

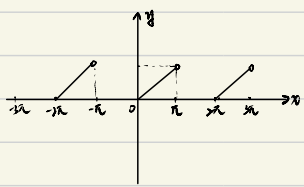
5° 令 $S(x)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$

例 2. $f(x)$ 以 2π 为周期，$f(x)$ 在 $[-\pi,\pi)$ 上表达式：

$$f(x)=\begin{cases}0 & -\pi\le x<0\\ x & 0\le x<\pi\end{cases}$$

展 $f(x)$ 为傅里叶，作和函数图像.

1° 作 $y=f(x)$ 图，找间断点.



$x=(2k+1)\pi$ $(k\in\mathbb{Z})$ 为 $f(x)$ 间断点，

2° $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{0}^{\pi}x\,dx=\frac{\pi}{2}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{0}^{\pi}x\cos nx\,dx=\frac{1}{n\pi}\int_{0}^{\pi}x\,d\sin nx$$

$$=\frac{1}{n\pi}x\sin nx\Big|_0^{\pi}-\frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx$$

$$=\frac{1}{n\pi}\cos nx\Big|_0^{\pi}=\frac{(-1)^n-1}{n\pi}=\begin{cases}-\frac{2}{n\pi} & n=1,3,5\cdots\\ 0 & n=2,4,6\cdots\end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx=\frac{1}{\pi}\int_0^{\pi}x\sin nx\,dx=-\frac{1}{n\pi}\int_0^{\pi}x\,d\cos nx$$

$$=-\frac{1}{n\pi}x\cos nx\Big|_0^{\pi}+\frac{1}{n\pi}\int_0^{\pi}\cos nx\,dx$$

$$=-\frac{(-1)^n\pi}{n\pi}=(-1)^{n+1}\cdot\frac{1}{n}$$

3° $f(x)=\frac{\pi}{4}+\left(-\frac{2}{\pi}\cdot1^{-1}\cos x+\sin x\right)$

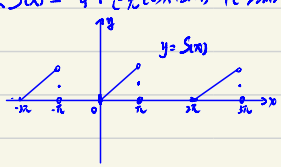
$+\left(-\frac{1}{2}\right)\sin 2x+\left(-\frac{1}{3\pi}\right)\cos 3x+\frac{1}{3}\sin 3x)$

$(-\infty<x<+\infty$ 且 $x\neq(2k+1)\pi$ $k\in\mathbb{Z})$

4° $x=(2k+1)\pi$，$k\in\mathbb{Z}$，时.

$f(x)\to\frac{\pi}{2}$

令 $S(x)=\frac{\pi}{4}+(-\frac{2}{\pi}(\cos x+\sin x)+(-\frac{1}{2}\sin 2x)+\cdots$



第十章　空间解析几何

Part 1 向量

一 def.

1. $\vec{a}$: 若 $|\vec{a}|=0$ → 零向量 $(\vec{0})$　　若 $\vec{a}\neq\vec{0}$,

$\vec{a}^\circ = \frac{\vec{a}}{|\vec{a}|}$

若 $|\vec{a}|=1$ → 单位向量

2. 向量坐标:

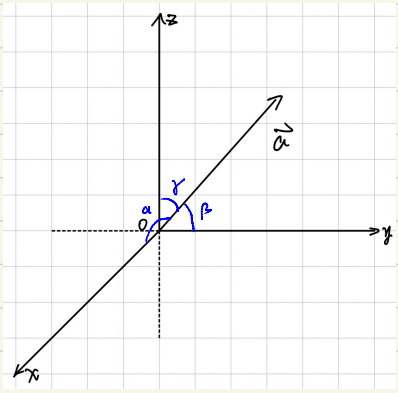
① 二维. $A(x_1,y_1)$　　$B(x_2,y_2)$

$\vec{AB} = (x_2-x_1)\vec{i}+(y_2-y_1)\vec{j} = \{x_2-x_1,\ y_2-y_1\}$

② 三维　$A(x_1,y_1,z_1)$　　$B(x_2,y_2,z_2)$

$\vec{AB} = (x_2-x_1)\vec{i}+(y_2-y_1)\vec{j}+(z_2-z_1)\vec{k} = \{x_2-x_1,\ y_2-y_1,\ z_2-z_1\}$.

☆ 3 方向角与方向余弦



设 $\vec{a}=\{a_1,b_1,c_1\}\neq\vec{0}$

$|\vec{a}| = \sqrt{a_1^2+b_1^2+c_1^2}$

2. α,β,γ – 方向角.

$\cos\alpha = \frac{a_1}{|\vec{a}|} = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$

$\cos\beta = \frac{b_1}{|\vec{a}|}$　　$\cos\gamma = \frac{c_1}{|\vec{a}|}$

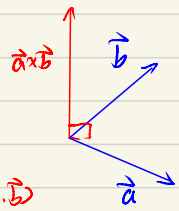
① $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

② $\{\cos\alpha,\cos\beta,\cos\gamma\} = \vec{a}^\circ$

二. 向量运算的几何刻划

① $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\langle\vec{a},\vec{b}\rangle$

② $\vec{a}\times\vec{b} = \begin{cases} \text{方向：右手准则} \\ \text{大小：} |\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\langle\vec{a},\vec{b}\rangle \end{cases}$



三 向量运算代数刻划

设 $\vec{a}=\{a_1, b_1, c_1\}$ $\vec{b}=\{a_2, b_2, c_2\}$

$\vec{a}\cdot\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

$\vec{a}\times\vec{b} = \begin{vmatrix} \cdot & \cdot & \cdot \end{vmatrix}$ → 做二阶行列式

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ $\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$

Notes:

1 $\vec{a}\cdot\vec{b}$:

① $\vec{a}\cdot\vec{a} = |\vec{a}|^2$, $\vec{a}\cdot\vec{a}=0 \Leftrightarrow \vec{a}=\vec{0}$

② $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

③ $\vec{a}\cdot\vec{b}=0 \Leftrightarrow a_1a_2+b_1b_2+c_1c_2 \Leftrightarrow \vec{a}\perp\vec{b}$

2. $\vec{a} \times \vec{b}$

① $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

② $\vec{a} \times \vec{b} \perp \vec{a}$, $\vec{a} \times \vec{b} \perp \vec{b}$

③ $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

④ $|\vec{a} \times \vec{b}| = 2S_{\triangle}$

例1. A $(-1, 0, 1)$ B $(1, 1, 2)$ C $(0, 1, -2)$ 求 $S_{\triangle ABC}$

解: $\vec{AC} = (1, 1, -3)$ $\vec{AB} = (2, 1, 1)$

$\vec{AB} \times \vec{AC} = \{-4, 7, 1\}$ $|\vec{AB} \times \vec{AC}| = \sqrt{66}$

$\Rightarrow S_{\triangle ABC} = \frac{1}{2}\sqrt{66}$

Part 2. 应用

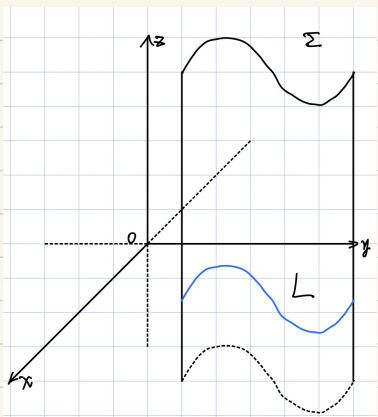
一. 空间曲面.

(一) 特殊曲面.

1. 柱面.

Σ: $F(x,y)=0$

L: $\begin{cases} F(x,y)=0 \\ z=0 \end{cases}$



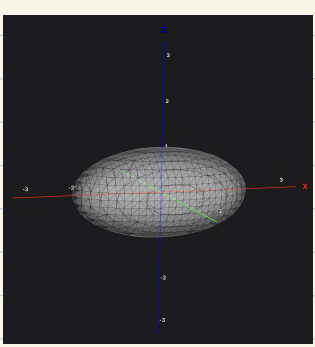
2. (2-dim) 旋转面

二维曲线绕坐标轴旋转.

$f(x,y)=0$ → 绕谁转, 谁不动

绕 x: $f(x,\pm\sqrt{y^2+z^2})=0$

绕 y: $f(\pm\sqrt{x^2+z^2},y)=0$



(三) 退化一平面

① 点法式.

$M_0\{x_0, y_0, z_0\} \in \pi$

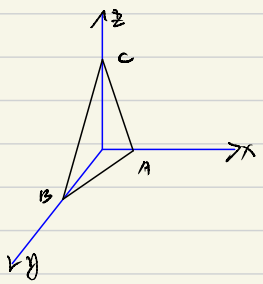
$\vec{n}\{A, B, C\} \perp \pi$ $\Rightarrow \pi: A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

② 截距式.

$\vec{AB}\{-a, b, 0\}$

$\vec{AC}\{-a, 0, c\}$

$\vec{n}=\vec{AB}\times\vec{AC}$



$=\{bc, ac, ab\}$ $\Rightarrow \pi = bc(x-a)+ac(y-0)+ab(z-0)$

$\Rightarrow \pi = \frac{x-a}{a}+\frac{y}{b}+\frac{z}{c}=0$

$\Rightarrow \pi: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

③ 一般式.

$\pi: Ax+By+Cz+D=0$

空间曲面

$F(x,y,z)=0$　　$M_0(x_0,y_0,z_0)\in\Sigma$

法向量　$\vec{n}=\{F_x,F_y,F_z\}_{M_0}$

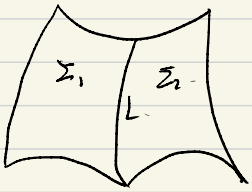
法平面　$F_x(x-x_0)+F_y(y-y_0)+F_z(z-z_0)=0$

二. 空间曲线

(一) 表示形式

1. 一般式

$$L:\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$$



2. 参数式

$$L:\begin{cases}x=R\cos\omega t\\y=R\sin\omega t\\z=vt\end{cases}$$

螺旋线

(二) 退化 - 直线.

1. 点向式.

$M_0(x_0, y_0, z_0) \in L$, $\vec{s}\{m, n, p\} \parallel L$

$L: \frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p}$

2. 参数式

令 $\frac{x-x_0}{m} = \frac{y-y_0}{n} = \frac{z-z_0}{p} = t$

$$L: \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

3. 一般式

$$L: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

例: $L: \begin{cases} x + y - 2 = 0 \\ 2x + y + z - 3 = 0 \end{cases}$ 化为点向式

取 $M_0(1, -1, 2) \in L$

$\vec{s} = \vec{n_1} \times \vec{n_2} = \{1, 1, 0\} \times \{2, 1, 1\} = \{1, -1, -3\}$

$L: \frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{-3}$

(三) 空间曲线 $\begin{cases} \text{切线} \\ \text{法平面} \end{cases}$

1. 参数式

$$L: \begin{cases} x=\varphi(t) \\ y=\phi(t) \\ z=\omega(t) \end{cases} \quad t=t_0 \Rightarrow \{\varphi(t_0), \phi(t_0), \omega(t_0)\}$$

$$\Rightarrow M_0(x_0, y_0, z_0) \in L.$$

$$\vec{T} = \{\varphi'(t_0), \phi'(t_0), \omega'(t_0)\} \quad \text{(切向量)}$$

2.

$$L: \begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \qquad M_0(x_0, y_0, z_0) \in L.$$

$$\vec{n_1} \{F_x, F_y, F_z\}\big|_{M_0}$$

$$\vec{n_2} \{G_x, G_y, G_z\}\big|_{M_0}$$

$$\Rightarrow \vec{T} = \vec{n_1} \times \vec{n_2}$$

三、距离

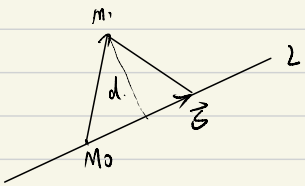
点与平面之距

$$\left.\begin{array}{l}\pi: Ax+By+Cz+D=0 \\ M_0(x_0, y_0, z_0) \notin \pi\end{array}\right\} \Rightarrow d=\frac{Ax_0+By_0+Cz_0+D}{\sqrt{A^2+B^2+C^2}}$$

平行平面之距

$$\left.\begin{array}{l}\pi_1: Ax+By+Cz+D_1=0 \\ \pi_2: Ax+By+Cz+D_2=0\end{array}\right\} d=\frac{|D_2-D_1|}{\sqrt{A^2+B^2+C^2}}$$

点到直线距离



$$|\overrightarrow{M_0M_1} \times \vec{s}| = 2\Delta S$$

$$|\vec{s}|\, d = 2\Delta S$$

$$\Rightarrow \quad d=\frac{|\overrightarrow{M_0M_1} \times \vec{s}|}{|\vec{s}|}$$

例2. $M_1(1,2,1)$ 求 M_1 到 L: $\begin{cases} x-y=0 \\ x+y-2z+1=0 \end{cases}$ 之距

解:

1° $M_0(1,1,1) \in L$

$\vec{s} = \{1,-1,0\} \times \{1,1,-2\} = \{1,1,2\}$

2° $\overrightarrow{M_0M_1} = (0,1,0)$

$\overrightarrow{M_0M_1} \times \vec{s} = \{2,0,-1\}$ $\therefore |\overrightarrow{M_0M_1} \times \vec{s}| = \sqrt{5}$

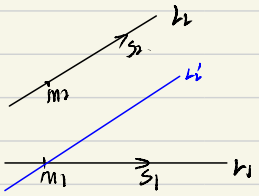
$|\vec{s}| = \sqrt{6}$ $\therefore d = \frac{\sqrt{30}}{6}$

异面直线之距

Notes

① L_1, L_2 共面 $\Leftrightarrow \vec{s_1} \times \vec{s_2} \perp \overrightarrow{M_1M_2}$

$\Leftrightarrow (\vec{s_1} \times \vec{s_2}) \cdot \overrightarrow{M_1M_2} = 0$



② L_1, L_2 异面 $\Leftrightarrow (\vec{s_1} \times \vec{s_2}) \cdot \overrightarrow{M_1M_2} \neq 0$

若 L_1, L_2 异面. 过 M_1 作 $L_2' \parallel L_2$. L_1 与 L_2' 共面于 π.

即 L_2 到 π 之距.

例3. $L_1: \frac{x}{1}=\frac{y-1}{0}=\frac{z}{1}$ $L_2: \begin{cases} xy=0 \\ xy-2z+2=0 \end{cases}$

解: 1° $M_1(0,1,0)$ $S_1(1,0,1)\in L_1$

$M_2(1,1,2)$ $S_2(2,2,2)\in L_2$

2° $\vec{S_1}\times\vec{S_2}=\{-2,0,2\}$

$\overrightarrow{M_1M_2}=\{1,0,2\}$ $(\vec{S_1}\times\vec{S_2})\cdot\overrightarrow{M_1M_2}\neq 0 \Rightarrow$ 异面.

3° 过M_1作 $L_2'\parallel L_2$. L_1与L_2'成平面π.

π: $-2(x-0)+0(y-1)+2(z-0)=0 \Rightarrow x-z=0$

4° $d=\frac{|1-2|}{\sqrt{1^2+0^2+(-1)^2}}=\frac{\sqrt{2}}{2}$.

第十一章
三重积分与曲线，曲面积分

三重积分

一 def ~ Ω 为有界几何体, $f(x,y,z)$ 在 Ω 上有界

1° $\Omega \Rightarrow \Delta V_1, \Delta V_2, \cdots \Delta V_n$;

2° $\forall (\xi_i, \eta_i, \zeta_i) \in \Delta V_i \quad (1 \le i \le n)$ 作

$$\sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta V_i;$$

3° 入为 $\Delta V_1, \Delta V_2, \cdots \Delta V_n$ 直径 max.

If $\lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta V_i$ 存在, 称此极限为 $f(x,y,z)$ 在 Ω 上的三重积分.

记 $\iiint_{\Omega} f(x,y,z)\,dV$.

$$\iiint_{\Omega} f(x,y,z)\,dV = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i, \zeta_i) \Delta V_i$$

二. 性质

1. $\iiint_{\Omega} 1\,dV = V$

2. ① Ω 关于 xOy 面对称 (关于立对称)

若 $f(x,y,-z) = -f(x,y,z) \Rightarrow \iiint_{\Omega} f\,dV = 0$

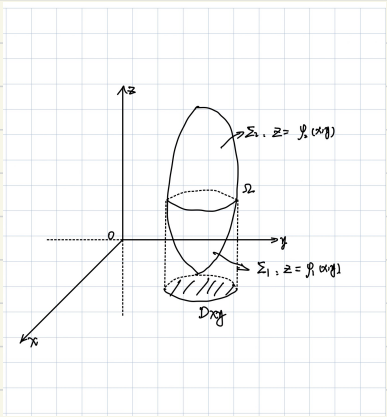
若 $f(x,y,-z) = f(x,y,z) \Rightarrow \iiint_{\Omega} f\,dV = 2\iiint_{\Omega_1} f\,dV$

三. 计算方法

(一) 直角坐标法

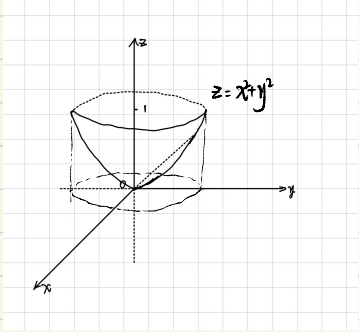
1. 铅直投影法

$$\Omega=\{(x,y,z)\mid (x,y)\in D_{xy},\ y_1(x,y)\le z\le y_2(x,y)\}$$



$$\iiint_\Omega f\,dV=\iint_{D_{xy}}dx\,dy\int_{y_1(x,y)}^{y_2(x,y)}f\,dz.$$

例1. $I=\iiint_\Omega z\,dV$



解: $D_{xy}: x^2+y^2\le 1$

$\Omega: \{(x,y,z)\mid (x,y)\in D_{xy},\ x^2+y^2\le z\le 1\}$

$$I=\iint_{D_{xy}}dx\,dy\int_{x^2+y^2}^{1}z\,dz.$$

$$=\frac{1}{2}\iint_{D_{xy}}[1-(x^2+y^2)^2]\,d\sigma$$

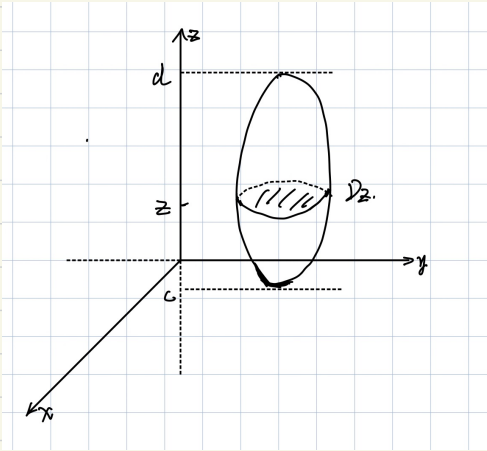
$$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^1 r-r^5\,dr$$

$$=\pi\left(\frac{1}{2}-\frac{1}{6}\right)=\frac{\pi}{3}.$$

2. 切片法

$$\Omega = \{(x,y,z) \mid (x,y) \in D_z,\ c \le z \le d\}$$

$$\iiint_\Omega f\,dV = \int_c^d dz \iint_{D_z} f(x,y,z)\,dx\,dy$$



例1. 为法：

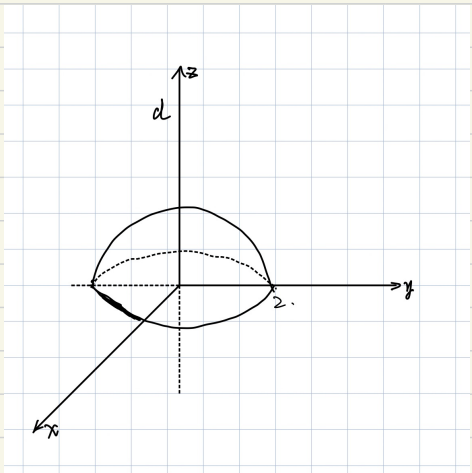
解：$\Omega = \{(x,y,z) \mid (x,y) \in D_z,\ 0 \le z \le 1\}$

$D_z: x^2+y^2 \le z$.

原式 $= \int_0^1 z\,dz \iint_{D_z} dx\,dy$

$= \pi \int_0^1 z^2\,dz = \frac{1}{3}\pi$.

例 2　$I=\iiint_\Omega x^2z\,dV$



解：$I=\iiint_\Omega y^2z\,dV$

$\Rightarrow 2I=\iiint_\Omega (x^2+y^2)z\,dV$

$\Rightarrow I=\frac{1}{2}\iiint_\Omega (x^2+y^2)z\,dV$

法 1.

D_{xy}　$x^2+y^2\le 4$

$\Omega=\{(x,y,z)\mid (x,y)\in D_{xy}$，

$0\le z\le \sqrt{4-x^2-y^2}\}$

$\therefore I=\frac{1}{2}\iint_{D_{xy}}(x^2+y^2)\,d\sigma\int_0^{\sqrt{4-x^2-y^2}}z\,dz$

$=\frac{1}{4}\iint_{D_{xy}}(x^2+y^2)(4-x^2-y^2)\,d\sigma$

$=\frac{1}{4}\int_0^{2\pi}d\theta\int_0^2 r^3(4-r^2)\,dr=\frac{\pi}{2}\cdot(16-\frac{32}{3})=\frac{\pi}{2}\times\frac{16}{3}$

$=\frac{8}{3}\pi$.

法二.

$D_z=x^2+y^2\le 4-z^2$

$I=\frac{1}{2}\int_0^2 z\,dz\iint_{D_z}(x^2+y^2)\,d\sigma$

$=\frac{1}{2}\int_0^2 z\,dz\int_0^{2\pi}d\theta\int_0^{\sqrt{4-z^2}}r^3\,dr$

$=\frac{\pi}{4}\int_0^2 z(4-z^2)^2\,dz=-\frac{\pi}{8}\int_0^2(4-z^2)^2\,d(4-z^2)=-\frac{\pi}{8}\cdot\frac{1}{3}(4-z^2)^3\Big|_0^2=\frac{8}{3}\pi$.

(二) 球面坐标变换法

1° 特征

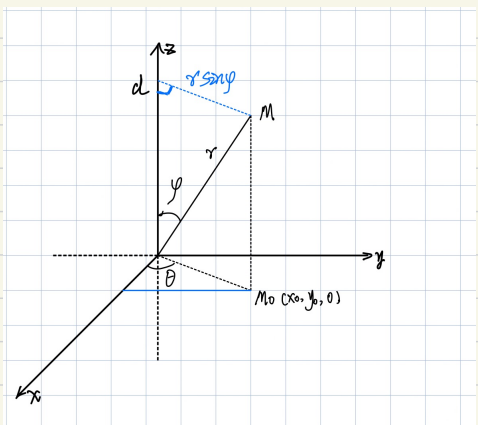
① Ω 边界曲面方程含 $x^2+y^2+z^2$;

② $f(x,y,z)$ 含 $x^2+y^2+z^2$

2° 变换

$M(r,\theta,\varphi)$ — 球坐标

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$



3°

$$dV = r^2\sin\varphi\, dr\, d\theta\, d\varphi$$

例2 法三.

$$I = \frac{1}{2}\iiint_{\Omega}(x^2+y^2)z\,dV$$

令: $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$ $(0\le\theta\le 2\pi,\ 0\le\frac{\pi}{2}\le\varphi,\ 0\le r\le 2)$

$$I = \frac{1}{2}\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^2 r^2\sin\varphi\cdot r^2\sin^2\varphi\cdot r\cos\varphi\,dr$$

$$= \pi\int_0^{\frac{\pi}{2}}\sin^3\varphi\,d\sin\varphi\int_0^2 r^5\,dr = \pi\times\frac{1}{4}\times\frac{64}{6} = \frac{8}{3}\pi.$$

例3. $\Omega: x^2+y^2+z^2 \le t^2 \quad (t>0)$ f 连续, $f(0)=0$, $f'(0)=2$

求 $\lim\limits_{t\to 0} \dfrac{\iiint_\Omega f(\sqrt{x^2+y^2+z^2})\,dv}{t^4}$.

解:

分子: $\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^t r^2 \sin\varphi \cdot f(r)\,dr$

$= 2\pi \int_0^{\pi} \sin\varphi\,d\varphi \int_0^t r^2 f(r)\,dr$

$= 4\pi \int_0^t r^2 f(r)\,dr$

原式 $= 4\pi \lim\limits_{t\to 0} \dfrac{t^2 f(t)}{4t^3} = \pi \lim\limits_{t\to 0} \dfrac{f(t)}{t} = \pi \lim\limits_{t\to 0} \dfrac{f(t)-f(0)}{t-0} = 2\pi$.

曲线积分与曲面积分

Part I 曲线积分

一、对弧长的曲线积分（第一类曲线积分）

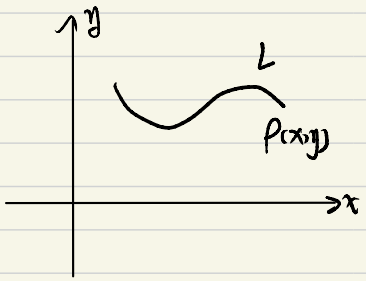
(一) 背景

① 线密度求 m

1° $\forall ds \subset L$

2° $dm = \rho(x,y)\,ds$

3° $m = \int_L dm = \int_L \rho(x,y)\,ds$



(二) 对弧长的曲线积分

$\int_L f(x,y)\,ds$ — $f(x,y)$ 在曲线段 L 上对弧长的曲线积分

(三) 性质

1. $\int_L 1\,ds = L$

2. ① L 左右对称，右 L_1

若 $f(-x,y) = -f(x,y) \Rightarrow \int_L f\,ds = 0$

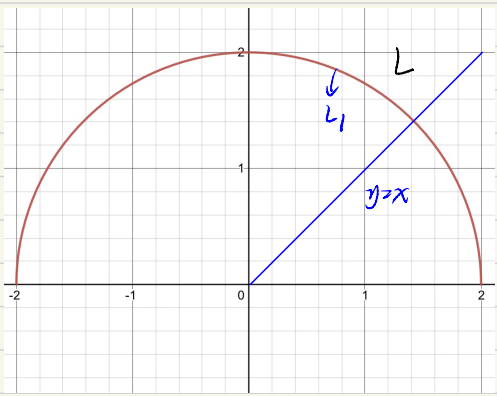
若 $f(-x,y) = f(x,y) \Rightarrow \int_L f\,ds = 2\int_{L_1} f\,ds$

② L 关于 $y=x$ 对称.

$$\int_L f(x,y)\,ds = \int_L f(y,x)\,ds$$

(四) 计算方法

例1. $I=\int_L (x^2-xy)\,ds$



解:

$$\because I=\int_L x^2\,ds$$

$$=2\int_{L_1} x^2\,ds$$

$$\because \int_{L_1} x^2\,ds=\int_{L_1} y^2\,ds$$

$$I=\int_{L_1}(x^2+y^2)\,ds$$

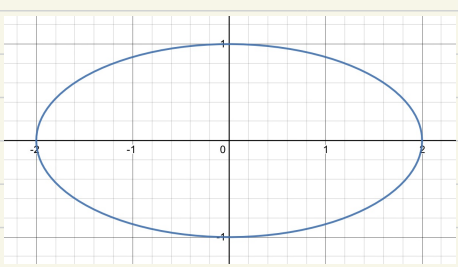
$$=\int_{L_1} 4\,ds = 4\cdot\frac{\pi}{2}\times 2 = 4\pi.$$

例2. $L=\frac{x^2}{4}+y^2=1$. L 长为 a, 求 $I=\int_L (x-2y)^2\,ds$

解: $I=\int_L (x^2-4xy+4y^2)\,ds$

$$=\int_L (x^2+4y^2)\,ds = 4\int_L (\frac{x^2}{4}+y^2)\,ds$$

$$=4a.$$



定积分法

对 $\int_L f(x,y)ds$

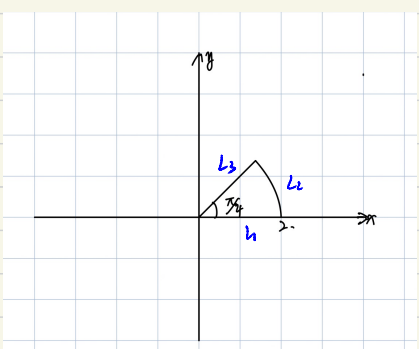
Case 1. $L: y=\varphi(x) \quad (a\le x\le b)$

$$\int_L f(x,y)ds = \int_a^b f[x,\varphi(x)]\sqrt{1+\varphi'^2(x)}\,dx$$

Case 2. $L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases} \quad (\alpha\le t\le\beta)$

$$\int_L f(x,y)ds = \int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt.$$

例3. $I=\int_L x e^{\sqrt{x^2+y^2}}ds$



解: $I=\int_{L_1}+\int_{L_2}+\int_{L_3}$

$L_1: y=0 \quad (0\le x\le 2)$

$\int_{L_1}=\int_0^2 x e^x\,dx$

$= xe^x\big|_0^2 - e^x\big|_0^2 = e^2+1$

$L_2:\begin{cases} x=2\cos t \\ y=2\sin t \end{cases} \quad (0\le t\le\frac{\pi}{4})$

$\int_{L_2}=\int_0^{\frac{\pi}{4}} 2\cos t\, e^2\sqrt{4\sin^2t+4\cos^2t}\,dt$

$= 4e^2\int_0^{\frac{\pi}{4}}\cos t\,dt = 2\sqrt{2}e^2$

$L_3: y=x \quad (0\le x\le\sqrt{2})$

$\int_{L_3}=\int_0^{\sqrt{2}} x e^{\sqrt{2}x}\sqrt{1^2+1^2}\,dx$

$=\frac{\sqrt{2}}{2}\int_0^{\sqrt{2}}\sqrt{2}x e^{\sqrt{2}x}\,d\sqrt{2}x$

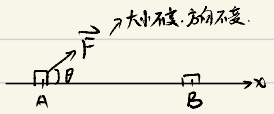
$=\frac{1}{\sqrt{2}}\int_0^2 xe^x dx = \frac{1}{\sqrt{2}}(x-1)e^x\big|_0^2 = \frac{1}{\sqrt{2}}(e^2+1)$

$\therefore I=(\frac{1}{\sqrt{2}}+1)(e^2+1)+2\sqrt{2}e^2$

二. 对坐标的曲线积分 (第二类曲线积分)

(一) 背景: 功

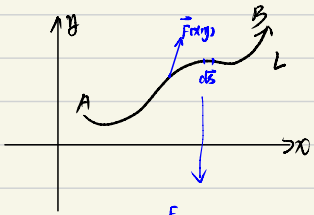
Case 1. (双理想)



$W = |\vec{F}|\cos\theta \cdot |\vec{AB}|$

$= \vec{F} \cdot \vec{AB}$

Case 2. (2-dim):



$\vec{F}(x,y) = \{P(x,y), Q(x,y)\}$.

1° $\forall\, d\vec{s} \subset L$. $d\vec{s} = \{dx, dy\}$.

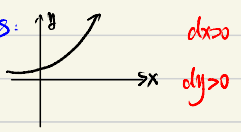
2° $dW = \vec{F} \cdot d\vec{s} = P(x,y)dx + Q(x,y)dy$

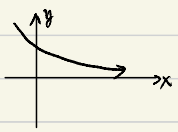
3°

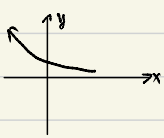
$$W = \int_L dW = \int_L P(x,y)dx + Q(x,y)dy$$



Notes:









Case 3 (3-dim):

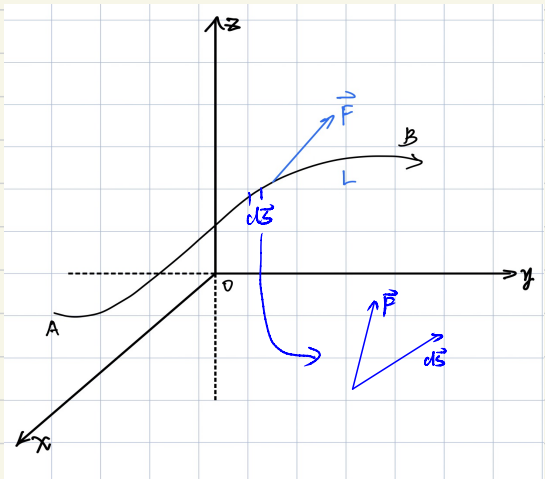
$\vec{F} = \{P, Q, R\}$.

1° $\forall d\vec{s} \subset L \quad d\vec{s} = \{dx, dy, dz\}$.

2° $dW = \vec{F} \cdot d\vec{s}$

$= P dx + Q dy + R dz$.

3° $W = \int_L P dx + Q dy + R dz$.



(二) 对坐标的曲线积分.

1. 2-dim $\int_L P(x,y)dx + Q(x,y)dy$

$\int_L P(x,y)dx$ - 函数 $P(x,y)$ 在有向曲线段 L 上对坐标 x 的曲线积分.

2. 3-dim. $\int_L P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$.

(三) 性质

1. $\int_{L^-} = -\int_L$

(四) 计算方法.

2-dim : $\int_L P(x,y)\,dx + Q(x,y)\,dy$

方法一：定积分法

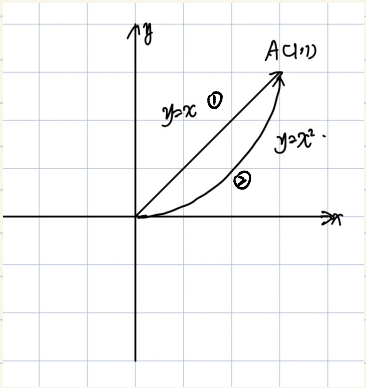
Case 1. L用直角坐标表示 - $y=\varphi(x)$ (起 $x=a$, 终 $x=b$) ↳方向

$$I=\int_a^b P[x,\varphi(x)]\,dx + Q[x,\varphi(x)]\,\varphi'(x)\,dx$$

Case 2. L用参数形式表示 - $L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$ (起 $t=\alpha$, 终 $t=\beta$)

$$I=\int_\alpha^\beta P[\varphi(t),\psi(t)]\,\varphi'(t)\,dt + Q[\varphi(t),\psi(t)]\,\psi'(t)\,dt.$$

例1. $I=\int_L (2y+1)\,dx + x^2\,dy$



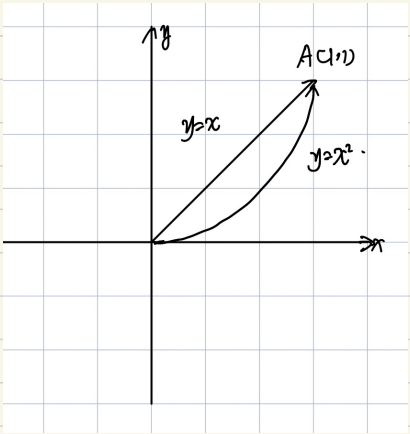
解：① L: $y=x$ (起 $x=0$, 终 $x=1$)

$$I=\int_0^1 (2x+1)\,dx + x^2\,dx$$
$$=\int_0^1 (x+1)^2\,dx = \frac{1}{3}(x+1)^3\Big|_0^1 = \frac{8}{3}$$

② L: $y=x^2$ (起 $x=0$, 终 $x=1$)

$$I=\int_0^1 (2x^2+1)\,dx + x^2\cdot 2x\,dx$$
$$=\int_0^1 2x^3+2x^2+1\,dx = \frac{1}{2}+\frac{2}{3}+1 = \frac{13}{6}$$

例2. $I=\int_L 2y\,dx+(2x+1)\,dy$



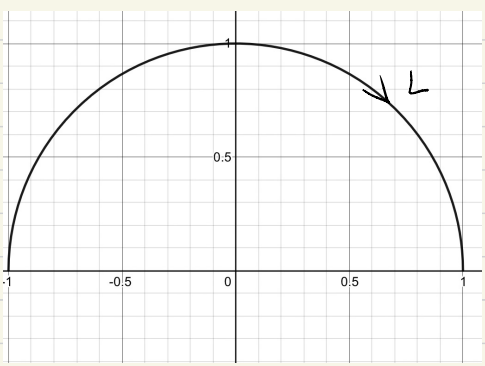
解：① $L: y=x$ (起 $x=0$，终 $x=1$)

$$I=\int_0^1 2x\,dx+(2x+1)\,dx$$
$$=\int_0^1 4x+1\,dx=2x^2+x\Big|_0^1=3$$

② $L: y=x^2$ (起 $x=0$，终 $x=1$)

$$I=\int_0^1 2x^2\,dx+(2x+1)2x\,dx$$
$$=\int_0^1 6x^2+2x\,dx=2x^3+x^2\Big|_0^1=3$$

例3. $I=\int_L y\,dx-\sqrt{x^2+y^2}\,dy$



解：

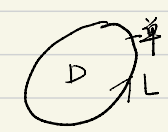
$$I=\int_L y\,dx-dy$$

$$L:\begin{cases}x=\cos t\\ y=\sin t\end{cases}\quad(\text{起}\ \pi,\ \text{终}\ 0)$$

$$I=\int_\pi^0 -\sin^2 t\,dt-\cos t\,dt$$
$$=\int_0^\pi \sin^2 t+\cos t\,dt$$
$$=2\int_0^{\frac{\pi}{2}}\sin^2 t\,dt+\sin t\Big|_0^\pi=2I_2=\frac{\pi}{2}.$$

方法二：二重积分法（Green 公式）

2-dim：联通区域



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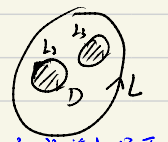


多外逆内顺正

D-二重积分　L-曲线积分

Th Green公式：

① 设 D为联通区域，L为D的正向边界

② $P(x,y)$，$Q(x,y)$ 在D上连续可偏导

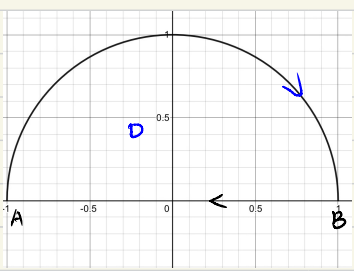
则 $\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$

例3. 为为.

解：$I = \int_L y\,dx - dy \Rightarrow P = y,\ Q = -1$　$\frac{\partial Q}{\partial x} = 0$　$\frac{\partial P}{\partial y} = 1$

由图可知

$I = \int_L = \oint_{L+\overline{BA}} + \int_{AB}$

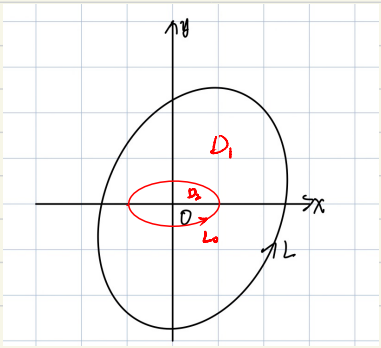


$\oint_{L+\overline{BA}} = -\iint_D (0-1)\,d\sigma = \iint_D d\sigma = \frac{\pi}{2}$

$\int_{AB} = \int_{-1}^{1} 0\,dx = 0$

∴ 原式 $= \pi/2$

例 4. 计算 $I=\oint_L \frac{x dy - y dx}{x^2+4y^2}$，L为绕O的正向闭曲线



解：① $P=-\frac{y}{x^2+4y^2}$　$Q=\frac{x}{x^2+4y^2}$

$\frac{\partial Q}{\partial x}=\frac{4y^2-x^2}{(x^2+4y^2)^2}$　$\frac{\partial P}{\partial y}=-\frac{x^2+4y^2}{(x^2+4y^2)^2}=\frac{4y^2-x^2}{(x^2+4y^2)^2}$

$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$　$[(x,y)\neq(0,0)]$ → Green不能用

(挖)

② 含 L_0：$x^2+4y^2=r^2$ 或 $\frac{x^2}{r^2}+\frac{y^2}{(\frac{r}{2})^2}=1$　($r>0$，L_0在L内，L_0逆时针)

② $\oint_{L+L_0}=\iint_{D_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)d\sigma=0$

$\Rightarrow \oint_L-\oint_{L_0}=0 \Rightarrow I=\oint_L=\oint_{L_0}$

④ $I=\oint_{L_0}\frac{x dy-y dx}{x^2+4y^2}=\oint_{L_0}\frac{x dy-y dx}{r^2}=\frac{1}{r^2}\oint_{L_0}x dy-y dx$

$=\frac{1}{r^2}\iint_{D_2}[1-(-1)]\,d\sigma=\frac{2}{r^2}\iint d\sigma=\frac{2}{r^2}\cdot\pi\cdot\frac{r}{2}=\pi$.

即原式为 π.

方法三：曲线积分与路径无关的条件

Th. ① D 为单联通区域

② $P(x,y)$，$Q(x,y)$ 在 D 内连续可偏导

以下等价：

① $\int_L P\,dx+Q\,dy$ 与路径无关；

② 任取闭曲线 $C\subset D$，$\oint_C P\,dx+Q\,dy=0$；

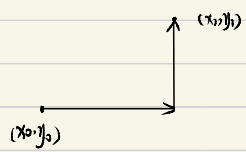
③ $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$（柯西-黎曼条件）

④ $\exists\, u(x,y)$ 使 $P\,dx+Q\,dy=du=\underbrace{\frac{\partial u}{\partial x}}_{P}dx+\underbrace{\frac{\partial u}{\partial y}}_{Q}dy$

Notes:

① If $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$，则 $\int_L P\,dx+Q\,dy=\int_{(x_0,y_0)}^{(x_1,y_1)}P\,dx+Q\,dy$

$=\int_{x_0}^{x_1}P(x,y_0)\,dx+\int_{y_0}^{y_1}P(x_1,y)\,dy$



② If $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$，且 $P\,dx+Q\,dy=du$

则 $\int_{(x_0,y_0)}^{(x_1,y_1)}P\,dx+Q\,dy=u(x_1,y_1)-u(x_0,y_0)$

③ If $\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$，$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}P\,dx+Q\,dy$

$=\int_{x_0}^{x}P(x,y_0)\,dx+\int_{y_0}^{y}Q(x,y)\,dy$

例1.

例3 设曲线积分$\int_L xy^2dx+y\varphi(x)dy$与路径无关，其中$\varphi$连续可导，且$\varphi(0)=2$，计算$\int_{(0,0)}^{(1,1)}xy^2dx+y\varphi(x)dy$.

djky68 (上岸)

解：1° $P=xy^2$, $Q=y\varphi(x)$ $\Rightarrow$ $y\varphi'(x)=2xy$

$\Rightarrow \varphi'(x)=2x$ $\Rightarrow \varphi(x)=x^2+C$ 而 $\varphi(0)=2$ $\therefore C=2$.

$\therefore \varphi(x)=x^2+2$

2°

法一：$I=\int_{(0,0)}^{(1,1)}xy^2dx+(x^2+2)ydy=\int_0^1 x\cdot 0^2dx+\int_0^1 3ydy=\frac{3}{2}$

法二：$xy^2dx+(x^2+2)ydy$

$=d(\frac{1}{2}x^2y^2)+d(y^2)=d(\frac{1}{2}x^2y^2+y^2)$

$\therefore I=\frac{1}{2}x^2y^2+y^2\Big|_{(0,0)}^{(1,1)}=\frac{3}{2}$

例2. 证 $x>0$ 内，$\frac{xdy-ydx}{x^2+y^2}$ 为某 $u(x,y)$ 全微分，求 $u(x,y)$

$P=\frac{-y}{x^2+y^2}$ $Q=\frac{x}{x^2+y^2}$

$\because \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}$

$\therefore \exists u(x,y)$ 使 $du=\frac{xdy-ydx}{x^2+y^2}$

$u(x,y)=\int_{(1,0)}^{(x,y)}\frac{xdy-ydx}{x^2+y^2}$

$=\int_1^x\frac{-0\,dx}{x^2+0^2}+\int_0^y\frac{xdy}{x^2+y^2}$

$=x\cdot\frac{1}{x}\operatorname{arctan}\frac{y}{x}\Big|_0^y=\operatorname{arctan}\frac{y}{x}$.

(三) 三维空间对曲线的坐标积分与 Stokes 公式

一、基本计算方法 —— 定积分法

↳ 一定要参数形式

设 $L:\begin{cases} x=\varphi(t) \\ y=\psi(t) \\ z=w(t) \end{cases}$ （起 $t=\alpha$，终 $t=\beta$）

$$\int_L P\,dx+Q\,dy+R\,dz=\int_\alpha^\beta P[\varphi(t),\psi(t),w(t)]\,\varphi'(t)\,dt+\cdots$$

例 1. $\int_L y\,dx-x\,dy+2xz\,dz$，$L:\begin{cases} x=t \\ y=-t \\ z=t^2 \end{cases}$ 从 $A(0,0,0)\to B(1,-1,1)$

解：原式 $=\int_0^1 -t^2\,dt-t(-2t)\,dt+2t^4\cdot 3t^2\,dt$

$=\int_0^1 (t^2+6t^6)\,dt=\frac{1}{3}+\frac{6}{7}=\frac{25}{21}$

例 2. $\int_L y\,dx-x\,dy+2\,dz$，$L:\begin{cases} x^2+y^2=1 \\ x-2y-z=2 \end{cases}$ 从 z 轴正向看逆时针.

解：1° $L=\begin{cases} x=\cos t \\ y=\sin t \\ z=\cos t-2\sin t-2 \end{cases}$ （起 $t=0$，终 $t=2\pi$）

2° 原式 $=\int_0^{2\pi} -\sin^2 t\,dt-\cos^2 t\,dt+2(-\sin t-2\cos t)\,dt$

$=\int_0^{2\pi} -1-2\sin t-4\cos t\,dt$

$=-2\pi+2\cos t\Big|_0^{2\pi}-4\sin t\Big|_0^{2\pi}=-2\pi$

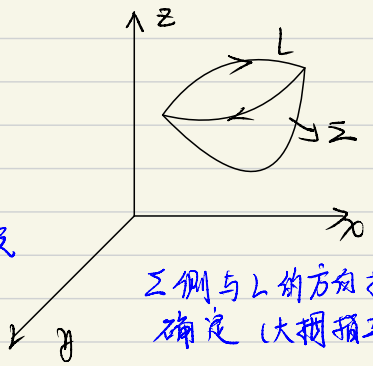
二. 曲面积分法 —— Stokes公式.

核心话题：域与界的关系

1-dim： $\int_a^b f(x)\,dx = F(b) - F(a)$

2-dim： $\oint_L P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$

3-dim： $\oiint_\Sigma P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$



几： Σ：光滑的曲面块

Γ：Σ的界

Σ侧与Γ方向按右手确定

P, Q, R 在Σ上连续可偏导

Σ侧与L的方向按右手确定（大拇指朝Σ方向）

则

$$\oint_\Gamma P\,dx + Q\,dy + R\,dz = \iint_\Sigma \begin{vmatrix} dy\,dz & dz\,dx & dx\,dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

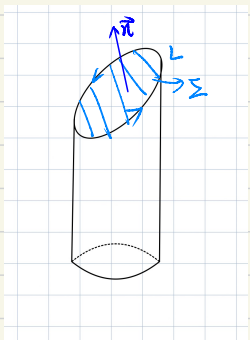
$$= \iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} ds$$

曲面Σ上法向量方向余弦

例 2. $\int_L y\,dx - x\,dy + 2dz$ L: $\begin{cases} x^2+y^2=1 \\ x-2y-z=2 \end{cases}$ z 轴正向逆时针

解:

1° Σ: $x-2y-z-2=0$



2° $\vec{n}$: ∵ 上侧 ⇒ γ 为锐角

∴ $\vec{n} = \{1, -1, -2\}$

$\cos\alpha = \frac{1}{\sqrt{6}}$ $\cos\beta = \frac{-2}{\sqrt{6}}$ $\cos\gamma = \frac{1}{\sqrt{6}}$

3° $\int_L y\,dx - x\,dy + 2dz$

$$= \frac{1}{\sqrt{6}} \iint_\Sigma \begin{vmatrix} -1 & -2 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 2 \end{vmatrix} ds$$

$(-1) \times (0) + (-2) \times (0) + 1 \times (-2) = -2$

$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial z}$ $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$ $\frac{\partial}{\partial z}$

y $-x$ 2 y $-x$ 2

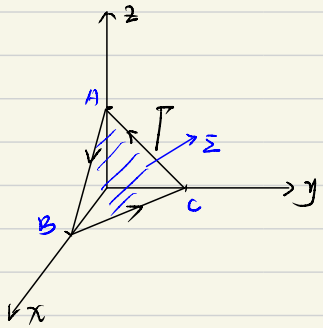
$$\Rightarrow -\frac{2}{\sqrt{6}} \iint_\Sigma 1\,ds$$

4° Σ: $z = x-2y-2$ $(x,y) \in D_{xy}$ D_{xy}: $x^2+y^2 \le 1$

$z'_x = 1$ $z'_y = -2$ ∴ $ds = \sqrt{1+4+4}\,d\sigma = \sqrt{6}\,d\sigma$

∴ 原式 $= -\frac{2}{\sqrt{6}} \iint_{D_{xy}} \sqrt{6}\,d\sigma = -2 \cdot \iint_{D_{xy}} 1\,d\sigma = -2\pi$

例 5. $\oint_\Gamma z\,dx + x\,dy + y\,dz$ Γ 为 $x+y+z=1$ 被三个坐标面所截三角形边界，从 z 轴正向逆时针.



1° $\Sigma: x+y+z-1=0$ 上侧

2° $\vec{n}: \{1,1,1\}$.

$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}}$

3°

$$I = \oint_\Gamma z\,dx + x\,dy + y\,dz$$

$$= \frac{1}{\sqrt{3}} \iint_\Sigma \begin{vmatrix} \frac{1}{\partial x} & \frac{1}{\partial y} & \frac{1}{\partial z} \\ z & x & y \end{vmatrix} ds = \sqrt{3} \iint_\Sigma 1\,ds$$

$\iint_\Sigma ds = |\vec{AB} \times \vec{AC}| \cdot \frac{1}{2}$

$\vec{AB} = (1,0,-1)$ $\vec{AC}(0,1,-1)$ $\therefore \iint_\Sigma ds = \frac{\sqrt{3}}{2}$

$\therefore$ 原式 $I = \frac{3}{2}$

二. 场论几个概念

1. 散度 — $\vec{A} = \{P(x,y,z), Q(x,y,z), R(x,y,z)\}$

$$\mathrm{div}\,\vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

2. 旋度 — $\vec{A} = \{P, Q, R\}$

$$\mathrm{rot}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

3. 流量 — $\vec{A}\{P, Q, R\}$，Σ 为有侧曲面

$$\Phi = \iint_{\Sigma} P\,dydz + Q\,dzdy + R\,dxdy$$

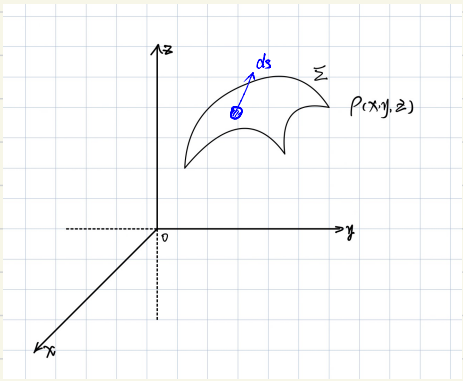
4. 环流量 — $\vec{A}\{P, Q, R\}$，Γ 为三维空间闭区线，有向

$$\Phi = \oint_{\Gamma} P\,dx + Q\,dy + R\,dz$$

Part 2 曲面积分

一 对面积的曲面积分 (第一类曲面积分)

(一) 背景



1° $\forall ds \subset \Sigma$;

2° $dm = \rho(x,y,z)\,ds$

3° $m = \iint\limits_{\Sigma} dm = \iint\limits_{\Sigma} \rho(x,y,z)\,ds$

(二) 对面积的曲面积分

$\iint\limits_{\Sigma} f(x,y,z)\,ds$ — $f(x,y,z)$ 在 Σ 上对面积的曲面积分.

(三) 性质

1. $\iint\limits_{\Sigma} 1\,ds = A$

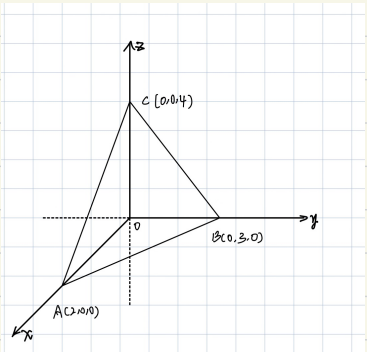
2. ① Σ 关于 xOy 面对称, 且 Σ_1

If $f(x,y,-z) = -f(x,y,z) \Rightarrow \iint\limits_{\Sigma} f\,ds = 0$

If $f(x,y,-z) = f(x,y,z) \Rightarrow \iint\limits_{\Sigma} f\,ds = 2\iint\limits_{\Sigma_1} f\,ds$

(四) 计算方法

例 1. $I = \iint_{\Sigma} (2x + \frac{4}{3}y + z)\,ds$



Σ: $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ $(x \geq 0,\ y \geq 0,\ z \geq 0)$

$I = 4 \iint_{\Sigma} (\frac{x}{2} + \frac{y}{3} + \frac{z}{4})\,ds$

$= 4 \iint_{\Sigma} ds$

$\vec{AB}$ $(-2, 3, 0)$ $\vec{AC}$ $(-2, 0, 4)$

$S_{\triangle ABC} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \sqrt{12^2 + 8^2 + 6^2}$

$= \frac{1}{2} \sqrt{244} = \sqrt{61}$

$\therefore I = 4\sqrt{61}$

二重积分法

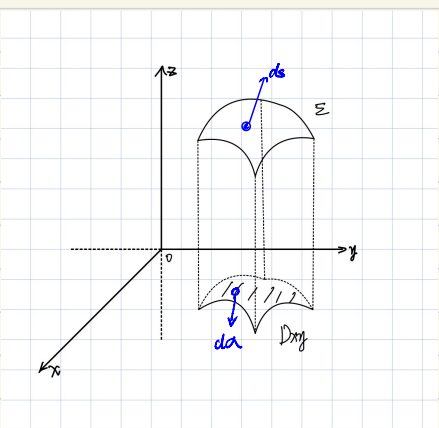
对 $\iint_{\Sigma} f(x, y, z)\,ds$

1° Σ: $z = \varphi(x, y)$ $(x, y) \in D_{xy}$

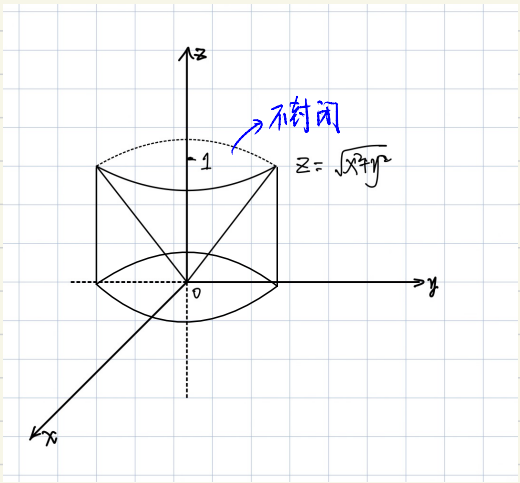
2° $\frac{\partial z}{\partial x} = ?$ $\frac{\partial z}{\partial y} = ?$

$ds = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,d\sigma$

3° $I = \iint_{D_{xy}} f[x, y, \varphi(x, y)] \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,d\sigma$



例 2. $I=\iint_{\Sigma}(z+xy)\,ds$



解：由对称性：

$I=\iint_{\Sigma} z\,ds$

Σ：$z=\sqrt{x^2+y^2}$　　$D_{xy}: x^2+y^2\le 1$

$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$　$\frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$

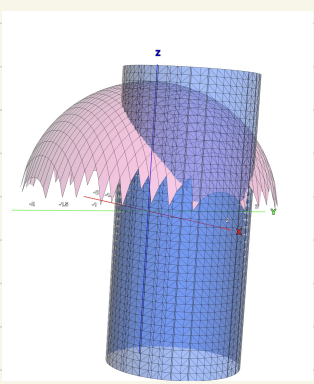
$ds=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,d\sigma=\sqrt{2}\,d\sigma$

$\therefore I=\iint_{D_{xy}}\sqrt{x^2+y^2}\,\sqrt{2}\,d\sigma=\sqrt{2}\int_0^{2\pi}d\theta\int_0^1 r^2\,dr$

$=2\sqrt{2}\pi\cdot\frac{1}{3}r^3\Big|_0^1=\frac{2}{3}\sqrt{2}\pi$.

例 3. $I=\iint_{\Sigma}(x^2z+xy)\,ds$　Σ 为 $z=\sqrt{4-x^2-y^2}$ 被柱面 $x^2+y^2=2x$ 所截.

解：由对称性：$I=\iint_{\Sigma} x^2z\,ds$



1° Σ：$z=\sqrt{4-x^2-y^2}$　　D_{xy}：$x^2+y^2\le 2x$

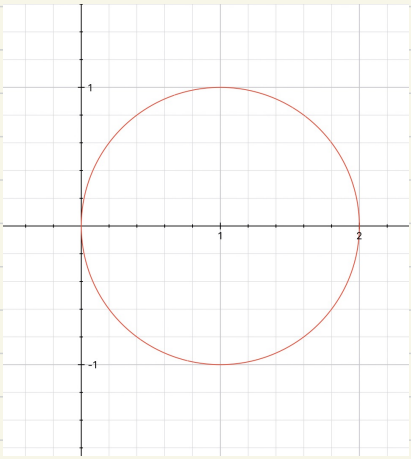
2° $\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{4-x^2-y^2}}$　$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{4-x^2-y^2}}$

3° $ds=\sqrt{1+\frac{x^2+y^2}{4-x^2-y^2}}=\frac{2}{\sqrt{4-x^2-y^2}}\,d\sigma$

4° $I=\iint_{D_{xy}} x^2\sqrt{4-x^2-y^2}\,\frac{2}{\sqrt{4-x^2-y^2}}\,d\sigma$

$= 2\iint_{D_{xy}} x^2 \, dA$

令 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\right)$



$\therefore I = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta \int_0^{2\cos\theta} r^3 \, dr$

$= 8\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\theta \, d\theta$

$= 16\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta = 16 I_6 = 16 \times \frac{5}{6} \times \frac{1}{4} \times \frac{\pi}{2} = \frac{5}{2}\pi$

二. 对坐标的曲面积分 (第二类曲面积分)

(一) 背景: 流量

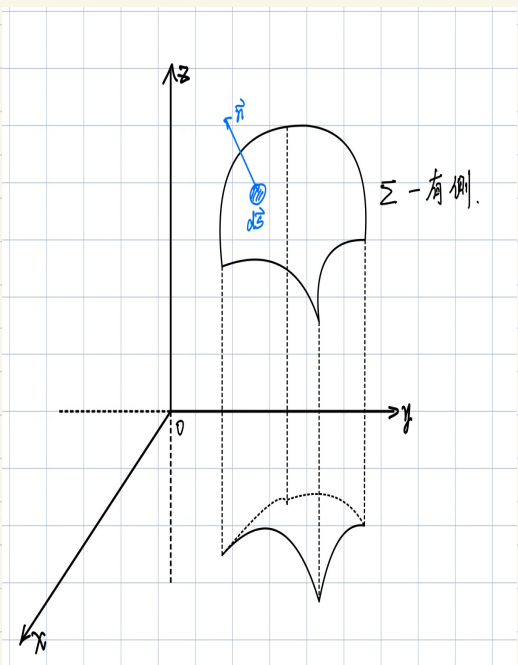
Note:

① $\forall d\vec{S} \subset \Sigma$

② 从 x 轴正方向, 将 $d\vec{S}$ 向 yoz 投影

记为 $dydz \begin{cases} >0 & \cos\alpha > 0 \\ <0 & \cos\alpha < 0 \end{cases}$

$dydz = \cos\alpha \cdot dS$



③ 从 y 轴正方向, 将 $d\vec{S}$ 向 xoz 投影

记为 $dxdz \begin{cases} >0 & \cos\beta > 0 \\ <0 & \cos\beta < 0 \end{cases}$

$dxdz = \cos\beta\, dS$

④ 从 z 轴正方向, 将 $d\vec{S}$ 向 xoy 投影

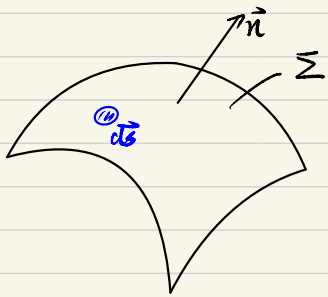
记为 $dxdy \begin{cases} >0 & \cos\gamma > 0 \\ <0 & \cos\gamma < 0 \end{cases}$

$dxdy = \cos\gamma\, dS$

$d\vec{S} = \{dydz,\ dzdx,\ dxdy\}$.

$= \{\cos\alpha dS,\ \cos\beta dS,\ \cos\gamma dS\}$

流体：$\vec{v}=\{P,Q,R\}$　Σ - 有侧曲面，则 $\Phi=?$



1° $\forall d\vec{s}\subset\Sigma$，$d\vec{s}=\{dydz, dzdx, dxdy\}$

2° $d\Phi=d\vec{s}\cdot\vec{v}$

$=Pdydz+Qdzdy+Rdxdy$

3° $\Phi=\iint\limits_{\Sigma}Pdydz+Qdzdy+Rdxdy$

(二) 对坐标的曲面积分

$\iint\limits_{\Sigma}Pdydz$ - $P(x,y,z)$ 在有侧曲面 Σ 上对坐标 y,z 的曲面积分.

(三) 性质

1. $\iint\limits_{\Sigma^-}=-\iint\limits_{\Sigma}$

2. $\iint\limits_{\Sigma}Pdydz+Qdzdx+Rdxdy=\iint\limits_{\Sigma}(P\cos\alpha+Q\cos\beta+R\cos\gamma)ds$

(四) 计算方法

方法1: 二重积分法

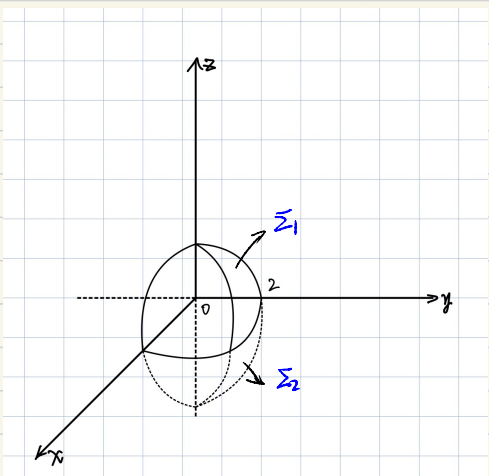
① $\iint\limits_{\Sigma} P(x,y,z)\,dy\,dz$:

1° Σ: $x=\varphi(y,z)$ $(y,z)\in D_{yz}$

2° $\iint\limits_{\Sigma} P\,\underbrace{dxdz}_{\text{投影,有}\pm} = \pm\iint\limits_{D_{yz}} P[\varphi(y,z),y,z]\,\underbrace{dydz}_{\text{面积}>0}$

$\cos\alpha>0$ 取正, $\cos\alpha<0$ 取负, 求前正, 后负.

例1. $I=\iint\limits_{\Sigma} z\,dxdy$



解: 1° $I=\iint\limits_{\Sigma_1} z\,dxdy+\iint\limits_{\Sigma_2} z\,dxdy$

2° Σ_1: $z=\sqrt{4-x^2-y^2}$

D_{xy}: $x^2+y^2\le 4$ $(x\ge 0, y\ge 0)$

$\iint\limits_{\Sigma_1} z\,dxdy=\iint\limits_{\Sigma_1}\sqrt{4-x^2-y^2}\,d\sigma$

Σ_2: $z=-\sqrt{4-x^2-y^2}$

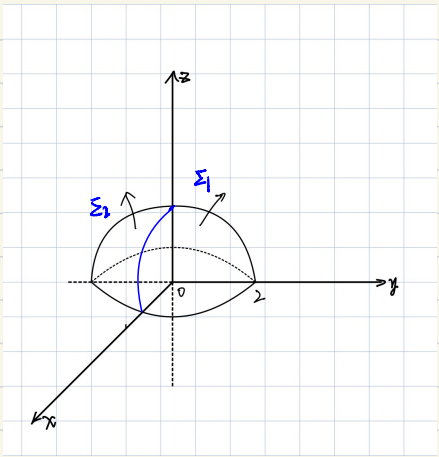
$\iint\limits_{\Sigma_2} z\,dxdy=-\iint\limits_{\Sigma_2}-\sqrt{4-x^2-y^2}\,d\sigma$

$=\iint\limits_{\Sigma_2}\sqrt{4-x^2-y^2}\,dxdy$.

3° $I=2\iint\limits_{D_{xy}}\sqrt{4-x^2-y^2}\,dxdy=2\int_0^{\frac{\pi}{2}}d\theta\int_0^2 r\sqrt{4-r^2}\,dr$

$=\frac{\pi}{2}\cdot\int_0^4\sqrt{4-t}\,dt=-\frac{\pi}{2}\cdot\frac{2}{3}(4-t)^{\frac{3}{2}}\Big|_0^4=\frac{8}{3}\pi$.

例2. $I=\iint_{\Sigma} yz\,dz\,dx+2\,dx\,dy$



解: $I=\iint_{\Sigma} yz\,dz\,dx+\iint_{\Sigma} 2\,dx\,dy$

$=I_1+I_2$

$I_1=\iint_{\Sigma} yz\,dz\,dx$

$=2\iint_{\Sigma_1} yz\,dz\,dx$

Σ_1: $y=\sqrt{4-x^2-z^2}$ $D_{xz}: x^2+z^2\le 4$ $(z\ge 0)$

$I_1=2\iint_{D_{xz}}\sqrt{4-x^2-z^2}\,z\,dx\,dz$

$=2\int_0^{\pi}\sin\theta\int_0^2\sqrt{4-r^2}\cdot r^2\,dr$

$=4\int_0^2\sqrt{4-r^2}\,r^2\,dr$ $=64\,(I_2-I_4)=4\pi$

$r=2\sin t$ $4\int_0^{\frac{\pi}{2}}4\sin^2t\,4(1-\sin^2t)\,dt$

Σ_2: $z=\sqrt{4-x^2-y^2}$, D_{xy} $x^2+y^2\le 4$

$I_2=2\iint dx\,dy$

$=8\pi$

Pq $I=12\pi$.

方法二：三重积分法（高斯公式）

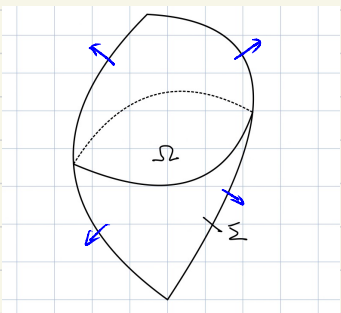
1-dim：$\int_a^b f(x)dx = F(b)-F(a)$

2-dim：$\oint_L P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$

3-dim

Σ：一曲面积分 - 外侧为正侧

Ω：三重积分



Th Gauss 公式

① Ω 为几何体，Σ 为 Ω 外表面

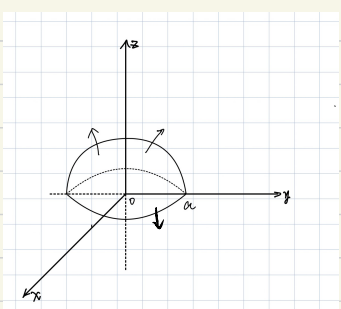
② P, Q, R 连续可偏导

则 $\oiint_{\Sigma} P dydz + Q dzdx + R dxdy$

$= \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$

例1 $I = \oiint_{\Sigma} xz^2 dydz + (x^2y - z^3) dzdx + (2xy + y^2z) dxdy$

Σ：$z = \sqrt{a^2 - x^2 - y^2}$ 与 $z \geq 0$ 围成的区域的外侧



解：$\frac{\partial P}{\partial x} = z^2$　$\frac{\partial Q}{\partial y} = x^2$　$\frac{\partial R}{\partial z} = y^2$

$I = \iiint_{\Omega} x^2 + y^2 + z^2 \, dV$

$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r^2 \cdot r^2 \sin\varphi \, dr$

$= 2\pi \cdot 1 \times \frac{a^5}{5} = \frac{2}{5}a^5\pi$

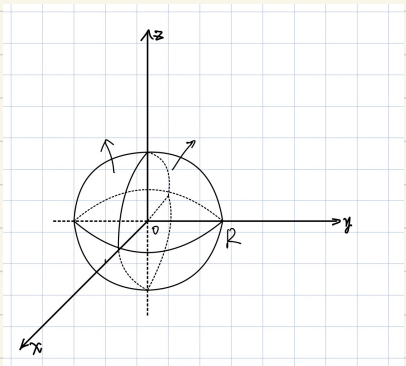
例 2. $I = \iint_{\Sigma} (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)\,dS$ $\Sigma: x^2+y^2+z^2=R^2$ 外侧

解: $I = \iint_{\Sigma} x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy$

$= 3\iiint_{\Omega} (x^2+y^2+z^2)\,dV$

$= 3\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{R} r^2 \cdot r^2\sin\varphi\,dr$

$= 6\pi \times 2 \times \frac{R^5}{5} = \frac{12}{5}\pi R^5$



例 3. $I = \iint_{\Sigma} \frac{ax\,dy\,dz + (z+a)^2\,dx\,dy}{\sqrt{x^2+y^2+z^2}}$ $(a>0)$

$\Sigma: z = -\sqrt{a^2-x^2-y^2}$ 上侧

解: 1° $I = \frac{1}{a}\iint_{\Sigma} ax\,dy\,dz + (z+a)^2\,dx\,dy = \frac{1}{a}I_1$

2° $P = ax \quad Q = 0 \quad R = (z+a)^2$

$\frac{\partial P}{\partial x} = a \quad \frac{\partial Q}{\partial y} = 0 \quad \frac{\partial R}{\partial z} = 2(z+a)$

3° $\Sigma_0: z=0 \quad (x^2+y^2 \le a^2)$ 下侧

$I_1 = \oiint_{\Sigma+\Sigma_0} - \iint_{\Sigma_0}$

$\oiint_{\Sigma+\Sigma_0} = -\iiint_{\Omega} (2z+3a)\,dV = -2\iiint_{\Omega} z\,dV - 3a\cdot\frac{2}{3}\pi a^3$

$= -2\int_0^{2\pi} d\theta \int_{\frac{\pi}{2}}^{\pi} d\varphi \int_0^{a} r^2\sin\varphi \cdot r\cos\varphi\,dr - 2\pi a^4$

$= 2\pi\cdot\frac{a^4}{4} - 2\pi a^4 = -\frac{3}{2}\pi a^4$

$\iint_{\Sigma_0} a^2\,dx\,dy = -\pi a^4$

$\therefore I = -\frac{1}{2}\pi a^3$